

Spatially asymptotic S-matrix from general boundary formulation

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We construct a new type of S-matrix in quantum field theory using the general boundary formulation. In contrast to the usual S-matrix the space of free asymptotic states is located at spatial rather than at temporal infinity. Hence, the new S-matrix applies to situations where interactions may remain important at all times, but become negligible with distance. We show that the new S-matrix is equivalent to the usual one in situations where both apply. This equivalence is mediated by an isomorphism between the respective asymptotic state spaces that we construct. We introduce coherent states that allow us to obtain explicit expressions for the new S-matrix. In our formalism crossing symmetry becomes a manifest rather than a derived feature of the S-matrix.

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I. INTRODUCTION

The general boundary formulation (GBF) [1, 2, 3] is a formulation of quantum theory that takes explicit account of spacetime. While in the standard formulation of quantum theory time does play an essential role, space only enters as a secondary concept as an ingredient of specific theories. Ironically, it is precisely this “more” in structure of going from time to spacetime that allows one to use “less” of it. Namely, while in the standard formulation the metric nature of time is essential, in the GBF only the topological structure of spacetime is indispensable. The latter turns out to be sufficient to formulate such fundamental concepts as probability conservation [2]. Hence, the GBF is naturally suitable to formulate quantum theories without metric background, such as quantum gravity. Furthermore, it allows to avoid other problems that arise in many approaches to quantizing gravity that are manifestly free of a background metric, such as the problem of locality. In quantum field theory the fact that we may choose to describe a given system only, without worrying about the rest of the universe, rests on principles such as causality and cluster decomposition. These in turn depend on the background metric of spacetime. Hence, without such a metric there is a priori no way to separate a system from the rest of the universe. In the GBF such a separation is achieved from the outset simply because one deals with the physics of spacetime regions that are explicitly separated from the rest of the universe through a boundary.

In the GBF state spaces that describe the objects of the theory are associated with boundaries of regions of spacetime (or components of these boundaries). One might think of such a state space as encoding the information that can potentially be exchanged between the region and the rest of the universe through the boundary. Amplitudes are associated with such regions and boundary states. These encode physical processes within the region and allow to calculate probabilities associated with measurements performed on the region. The standard state spaces and transition amplitudes are recovered when the spacetime region in question is a time interval times all of space. The boundary state space then splits into a tensor product of the initial state space with the final one, associated to initial and final boundary components respectively. The amplitude corresponding to this region becomes just the usual transition amplitude.

To emphasize it again, the GBF is merely a theoretical framework and not in itself any specific quantum theory. Rather, it is a specific definition of what constitutes a quantum theory and how predictions are extracted in principle from its ingredients. What kind of spacetime structures (regions and hypersurfaces) enter into the GBF depends on the theory under consideration. The minimum required are topological manifolds of a given dimension, for an example of this type see [4]. Slightly more structure would be provided by differentiable manifolds. One might reasonably assume that this is the relevant geometric setting for a quantum theory of gravity. Lorentzian manifolds and, more specifically, submanifolds of Minkowski space are the relevant ingredient for ordinary quantum field theory. For conformal field theory one would have manifolds with conformal structure etc.

Providing a viable framework for quantum gravity is one of the main motivations for the GBF. The program associated with this goal consists of establishing and developing the GBF in the context of known and tested quantum

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theories as well as with simplified models incorporating key features expected of a quantum theory of gravity. A key conjecture of this program is that ordinary quantum field theory can be extended to fit into this framework for a sufficiently interesting class of admissible spacetime regions and hypersurfaces. We refer to this in the following as the *extensibility conjecture*. That quantum field theory fits into the GBF if we restrict to equal-time hyperplanes and time-interval regions is essentially trivial. The emphasis is hence on the word “interesting”. While it is technically difficult to deal with generic hypersurfaces and regions, even very special and highly symmetric geometries can serve to demonstrate the power of the GBF if they differ from the standard geometry in sufficiently radical ways.

For example, an important restriction of the standard framework is that state spaces are only ever associated to spacelike (Cauchy) hypersurfaces. The more technical reason for this is that canonical quantization prescriptions usually rely on specific correspondences in the classical theory between data on such hypersurfaces and solutions in all of spacetime. The more conceptual reason for this is that in the probability interpretation of transition amplitudes we are used to think in terms of a strict temporal ordering between the prepared and the observed states. This might lead one to suspect that serious problems appear when trying to formulate the quantum theory with more general hypersurfaces, namely hypersurfaces that contain timelike parts. It was shown in [5] that these objections are unfounded. More precisely, it was shown for the Klein-Gordon quantum field theory how all the ingredients of the theory including vacuum, full state space, amplitudes etc. can be consistently defined on *timelike* hyperplanes and associated interval-like regions. In crucial difference to the spacelike case, a state can not be generally labeled as incoming or outgoing. Rather each particle in a general multi-particle state may individually be incoming or outgoing.

This example still shares an important feature with the standard framework. Although the temporal ordering between the participating states is lost, an amplitude can still be thought of as a *transition* amplitude between one state and another. In geometric terms this is because the region with which the amplitude is associated has a boundary which decomposes into two connected components, namely the hypersurfaces which carry the states. A further step was hence taken in [6] where it was shown explicitly that the division into two state spaces associated to different boundary components can be given up. The geometry used was the solid hypercylinder $\mathbb{R} \times B_R^3$, i.e., a three-ball of radius R in space extended over all of time. The boundary $\mathbb{R} \times S_R^2$ of this region is *connected*, implying that the associated state space does not decompose into a tensor product of components associated to subspaces of this boundary. Nevertheless, physical sense can be made of amplitudes associated with states on this hypercylinder and probabilities be extracted.

The extensions of quantum field theory described so far were only elaborated in the free theory. This is not surprising. Even in the standard framework we mostly do not know what the state space of an interacting quantum field theory is. However, we have a highly successful technique which allows us to say a lot about interacting quantum field theory without this knowledge: The S-matrix. Assuming that interactions are only relevant at intermediate times, one considers the transition amplitude between free states at an initial time and free states at a final time. One then takes the asymptotic limit of this amplitude where the initial time goes to $-\infty$ and the final time goes to $+\infty$. This yields the S-matrix.

A convincing argument for the validity of the extensibility conjecture in interacting quantum field theory can be made, if we manage to construct an asymptotic amplitude starting from an interesting non-conventional geometry and show its physical equivalence to the S-matrix. This is precisely what we announced in a previous letter [7] and detail in this paper. The geometry we choose is that of the aforementioned hypercylinder $\mathbb{R} \times S^2$ as it is not only timelike, but also connected and exhibits most of the new features of the GBF in this context. Physically, the starting assumption is that interactions can be neglected outside a certain finite region of space. However, no assumption needs to be made about interactions being negligible at any specific time. We consider then the amplitude associated with the solid hypercylinder $\mathbb{R} \times B_R^3$ of sufficiently large radius R for a free state on its boundary $\mathbb{R} \times S_R^2$. The sought for asymptotic amplitude arises in the limit $R \rightarrow \infty$. We go on to show that there is an isomorphism of Hilbert spaces identifying the usual (temporal) asymptotic state space of the S-matrix with the (spatial) asymptotic state space of the hypercylinder in the infinite radius limit. Under this mapping the S-matrix and the asymptotic hypercylinder amplitude become identical.

The paper is organized as follows. In Section II we recall some basic aspects of the massive Klein-Gordon field in Minkowski spacetime. In particular, we use the Feynman path integral quantization procedure combined with the Schrödinger representation and coherent states. In Section III the S-matrix elements in the basis of the coherent states are computed in three steps: we first consider the free theory, then we study the interaction of the scalar field with an external source, and finally functional methods allow us to work out the S-matrix for the general interacting theory. While the resulting S-matrix is well known, the purpose of our derivation is to serve as a blueprint for the novel case of the asymptotic amplitude on the hypercylinder that is treated later. Also, we have not found the derivation in this form elsewhere and it might thus have some interest of its own. In Section IV we treat the Klein-Gordon field defined on an hypercylinder within the GBF. We recall the main features of the classical and the quantum theories studied in [6] adapting them to the needs of the present setting. We then proceed to introduce coherent states on the hypercylinder. In Section V the asymptotic amplitude for the hypercylinder is derived in analogy to the same three

steps performed in Section III. In Section VI we compare the expressions of the two asymptotic amplitudes obtained, showing them to be identical under a suitable isomorphism of the asymptotic state spaces. We end by presenting conclusions and an outlook in Section VII. Appendix A describes the conversion of coherent states from the Fock representation to the Schrödinger representation. Appendix B contains useful formulas for spherical harmonics and different types of Bessel functions.

II. FREE KLEIN-GORDON FIELD IN MINKOWSKI SPACETIME

We start with the real massive Klein-Gordon quantum field theory in Minkowski spacetime whose action in a region M is given by

$$S_{M,0}(\phi) = \frac{1}{2} \int_M d^4x \left((\partial_0 \phi)(\partial_0 \phi) - \sum_{i \geq 1} (\partial_i \phi)(\partial_i \phi) - m^2 \phi^2 \right). \quad (1)$$

(The index 0 indicates that we consider the free theory.) The equation of motion is the Klein-Gordon equation

$$(\square + m^2) \phi = 0, \quad (2)$$

where $\square = \partial_0^2 - \sum_{i \geq 1} \partial_i^2$.

A. Schrödinger-Feynman approach

We use the Schrödinger representation for quantum states. That is, quantum states are wave functions on the space of instantaneous field configurations. While this is the usual approach in non-relativistic quantum theory it is less frequently used in quantum field theory, but see [8, 9]. At the same time we use the Feynman path integral to represent transition amplitudes. In contrast to other quantization prescriptions, such as canonical quantization, this setting generalizes readily to the general boundary formulation, which we will use starting from Section IV.

We recall below basic elements of the Schrödinger-Feynman setting applied to the Klein-Gordon quantum field theory. The inner product of states is,

$$\langle \psi_2 | \psi_1 \rangle = \int \mathcal{D}\varphi \psi_1(\varphi) \overline{\psi_2(\varphi)}, \quad (3)$$

where the integral is over all field configurations.

The transition amplitude from the state of the system at time t_1 described by the wave function $\psi_1(\varphi_1)$ to the state of the system at time t_2 described by the wave function $\psi_2(\varphi_2)$ takes the form

$$\langle \psi_2 | U_{[t_1, t_2]} | \psi_1 \rangle = \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \psi_1(\varphi_1) \overline{\psi_2(\varphi_2)} Z_{[t_1, t_2]}(\varphi_1, \varphi_2), \quad (4)$$

where $U_{[t_1, t_2]}$ represents the time-evolution operator, and the field propagator is given by

$$Z_{[t_1, t_2]}(\varphi_1, \varphi_2) = \int_{\substack{\phi|_{t_1}=\varphi_1 \\ \phi|_{t_2}=\varphi_2}} \mathcal{D}\phi e^{iS_{[t_1, t_2]}(\phi)}. \quad (5)$$

In the case of the free theory determined by the action (1) we can evaluate the associated propagator $Z_{[t_1, t_2], 0}$ by shifting the integration variable by a classical solution ϕ_{cl} that interpolates between φ_1 at t_1 and φ_2 at t_2 . Although the result can be readily found in [9] or [6], we repeat the derivation here since it provides an instructive example for calculations in later sections that follow the same pattern. Explicitly,

$$Z_{[t_1, t_2], 0}(\varphi_1, \varphi_2) = \int_{\substack{\phi|_{t_1}=\varphi_1 \\ \phi|_{t_2}=\varphi_2}} \mathcal{D}\phi e^{iS_{[t_1, t_2], 0}(\phi)} = \int_{\substack{\phi|_{t_1}=0 \\ \phi|_{t_2}=0}} \mathcal{D}\phi e^{iS_{[t_1, t_2], 0}(\phi_{cl} + \phi)} = N_{[t_1, t_2], 0} e^{iS_{[t_1, t_2], 0}(\phi_{cl})}, \quad (6)$$

where the normalization factor is formally given by

$$N_{[t_1, t_2], 0} = \int_{\substack{\phi|_{t_1}=0 \\ \phi|_{t_2}=0}} \mathcal{D}\phi e^{iS_{[t_1, t_2], 0}(\phi)}. \quad (7)$$

ϕ_{cl} can be decomposed into positive and negative energy modes as

$$\phi_{cl}(x, t) = e^{-i\omega t} \varphi^+(x) + e^{i\omega t} \varphi^-(x). \quad (8)$$

Here the time variable t belongs to the interval $[t_1, t_2]$ and ω is the operator

$$\omega := \sqrt{-\sum_{i \geq 1} \partial_i^2 + m^2}. \quad (9)$$

The configurations φ^\pm are related to the configurations on the boundary, φ_1 for $t = t_1$ and φ_2 for $t = t_2$, by the relation

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} e^{-i\omega t_1} & e^{i\omega t_1} \\ e^{-i\omega t_2} & e^{i\omega t_2} \end{pmatrix} \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}. \quad (10)$$

We invert the matrix to express the configurations φ^\pm as function of the configurations φ_1 and φ_2 :

$$\begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} = \frac{1}{2i \sin \omega(t_2 - t_1)} \begin{pmatrix} e^{i\omega t_2} & -e^{i\omega t_1} \\ -e^{-i\omega t_2} & e^{-i\omega t_1} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \quad (11)$$

Substituting this into (8), we get the expression for the classical solution ϕ_{cl} in terms of φ_1 and φ_2 ,

$$\phi_{cl}(t, x) = \frac{\sin \omega(t_2 - t)}{\sin \omega(t_2 - t_1)} \varphi_1(x) + \frac{\sin \omega(t - t_1)}{\sin \omega(t_2 - t_1)} \varphi_2(x). \quad (12)$$

This allows in turn to evaluate the field propagator,

$$Z_{[t_1, t_2], 0}(\varphi_1, \varphi_2) = N_{[t_1, t_2], 0} \exp \left(-\frac{1}{2} \int d^3x \begin{pmatrix} \varphi_1 & \varphi_2 \end{pmatrix} W_{[t_1, t_2]} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right), \quad (13)$$

where $W_{[t_1, t_2]}$ is the operator valued 2×2 matrix

$$W_{[t_1, t_2]} = \frac{-i\omega}{\sin \omega(t_2 - t_1)} \begin{pmatrix} \cos \omega(t_2 - t_1) & -1 \\ -1 & \cos \omega(t_2 - t_1) \end{pmatrix}. \quad (14)$$

The vacuum wave function is,

$$\psi_0(\varphi) = C \exp \left(-\frac{1}{2} \int d^3x \varphi(x) (\omega \varphi)(x) \right), \quad (15)$$

where C is a normalization factor.

B. Coherent states

Pioneered by Glauber [10] to study electromagnetic correlation functions in quantum optics, coherent state techniques are by now widely used in quantum field theory and particle physics. The interest in coherent states comes mainly from the possibility to construct quantum states that correspond, in the limit where the number of field quanta is large, to classical field configurations. We define coherent states for the Klein-Gordon theory in Minkowski spacetime following the conventions in [11]. We first adopt a Fock representation to express the coherent states and then pass to the Schrödinger representation.

Indicating the vacuum state of the scalar field in the Fock representation with $|0\rangle$, a normalized coherent state takes the form

$$|\psi_\eta\rangle = C_\eta \exp \left(\int \frac{d^3k}{(2\pi)^3 2E} \eta(k) a^\dagger(k) \right) |0\rangle, \quad (16)$$

where $a^\dagger(k)$ is the creation operator for the field mode of momentum k . η is a complex function on momentum space that characterizes the state. The normalization constant C_η is given by

$$C_\eta = \exp \left(-\frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2E} |\eta(k)|^2 \right) \quad (17)$$

so that the inner product of coherent states is given by

$$\langle \psi_{\eta_2} | \psi_{\eta_1} \rangle = \exp \left(\int \frac{d^3 k}{(2\pi)^3 2E} \left(\eta_1(k) \overline{\eta_2(k)} - \frac{1}{2} |\eta_1(k)|^2 - \frac{1}{2} |\eta_2(k)|^2 \right) \right). \quad (18)$$

We can write the resolution of the identity operator I in terms of coherent states as

$$D^{-1} \int d\eta d\bar{\eta} |\psi_\eta\rangle \langle \psi_\eta| = I, \quad (19)$$

with

$$D = \int d\eta d\bar{\eta} \exp \left(- \int \frac{d^3 k}{(2\pi)^3 2E} |\eta(k)|^2 \right). \quad (20)$$

The time evolution of a coherent state is given by

$$e^{-iH\Delta t} |\psi_\eta\rangle = |\psi_{\eta'}\rangle \quad \text{with} \quad \eta'(k) = e^{-iE\Delta t} \eta(k). \quad (21)$$

The expansion of a coherent state in terms of multi-particle states can be read off directly from (16),

$$|\psi_\eta\rangle = C_\eta \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^3 k_1}{(2\pi)^3 2E_1} \cdots \int \frac{d^3 k_n}{(2\pi)^3 2E_n} \eta(k_1) \cdots \eta(k_n) |\psi_{k_1, \dots, k_n}\rangle \quad (22)$$

In particular, the inner product between a coherent state determined by η and a state with particles of momenta k_1, \dots, k_n is,

$$\langle \psi_{k_1, \dots, k_n} | \psi_\eta \rangle = C_\eta \eta(k_1) \cdots \eta(k_n). \quad (23)$$

As shown in Appendix A the expression for the coherent state in the Schrödinger representation is

$$\psi_\eta(\varphi) = K_\eta \exp \left(\int \frac{d^3 x d^3 k}{(2\pi)^3} \eta(k) e^{ikx} \varphi(x) \right) \psi_0(\varphi), \quad (24)$$

with $\psi_0(\varphi)$ being the wave function of the vacuum state. The normalization factor K_η , different from the normalization factor C_η in the Fock representation is given by

$$K_\eta = \exp \left(- \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3 2E} (\eta(k)\eta(-k) + |\eta(k)|^2) \right). \quad (25)$$

III. CONVENTIONAL S-MATRIX

The S-matrix is the standard tool to calculate probabilities of scattering processes in quantum field theory. One assumes that interactions can be neglected at very early and at very late times, where states are typically considered as consisting of a collection of free particles. Thus, to model these particles one uses the state space of the theory without interaction. Transition amplitudes between such free states at initial time t_1 and at final time t_2 can be calculated with the interaction switched on at intermediate times. The S-matrix is then the asymptotic limit of these transition amplitudes for $t_1 \rightarrow -\infty$ and $t_2 \rightarrow +\infty$.

In this section we compute the elements of the S-matrix in three different cases. First we consider the free Klein-Gordon theory, then we study the interacting theory where the interaction is given by a source term, and we finally treat general interactions by means of functional derivatives of the result obtained with the source interaction.

Although the result is standard, our derivation differs from standard textbook treatments. What is more, it provides a blueprint to the subsequent derivation of a new type of S-matrix in Section V. Our treatment is similar in spirit to the one using the holomorphic representation [12]. However, instead of dealing with propagation kernels we directly deal with transition amplitudes, using the coherent states of Section II B.

A. Free theory

In order to make sense of the limiting procedure involved in defining the S-matrix we switch to the interaction (or Dirac) picture. That is, we identify states at different times if they are related by time evolution in the free theory. To make this more transparent we add a label t to the state when appropriate, specifying at which time it is evaluated. Recalling the free time evolution (21) we obtain the wave function

$$\psi_{t,\eta}(\varphi) = K_{t,\eta} \exp \left(\int \frac{d^3x d^3k}{(2\pi)^3} \eta(k) e^{-i(Et-kx)} \varphi(x) \right) \psi_0(\varphi), \quad (26)$$

for the coherent state at time t which at time 0 coincides with (16). Note that the normalization factor (25) is now also time dependent and takes the form

$$K_{t,\eta} = \exp \left(-\frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2E} (e^{-2iEt} \eta(k) \eta(-k) + |\eta(k)|^2) \right). \quad (27)$$

The transition amplitude in the free theory in terms of interaction picture states is by construction independent of the initial and final time and equal to the inner product (18). In particular, we can let the initial time go to $-\infty$ and the final time go to $+\infty$ and think of these transition amplitudes as the elements of the S-matrix \mathcal{S}_0 of the free theory,

$$\langle \psi_{\eta_2} | \mathcal{S}_0 | \psi_{\eta_1} \rangle = \lim_{\substack{t_1 \rightarrow -\infty \\ t_2 \rightarrow +\infty}} \langle \psi_{t_2, \eta_2} | U_{[t_2, t_1], 0} | \psi_{t_1, \eta_1} \rangle = \langle \psi_{\eta_2} | \psi_{\eta_1} \rangle. \quad (28)$$

Here, we have denoted with $U_{[t_2, t_1], 0}$ the evolution operator of the free theory between the times t_1 and t_2 .

B. Theory with source

We consider in this section the Klein-Gordon theory interacting with a source field μ . In a spacetime region M the new action takes the form

$$S_{M,\mu}(\phi) = S_{M,0}(\phi) + \int_M d^4x \mu(x) \phi(x), \quad (29)$$

where $S_{M,0}$ is the free action (1). Here, M will be determined by a time interval $[t_1, t_2]$ and we assume the source field to vanish outside this time interval.

The path integral (5) determining the propagator $Z_{[t_1, t_2], \mu}$ for the theory with source can be evaluated in the same way (6) as in the free theory. That is, we shift the integration variable by a classical solution ϕ_{cl} of the free theory (not the one with source) interpolating between φ_1 at t_1 and φ_2 at t_2 to obtain

$$Z_{[t_1, t_2], \mu}(\varphi_1, \varphi_2) = N_{[t_1, t_2], \mu} e^{iS_{[t_1, t_2], \mu}(\phi_{\text{cl}})}. \quad (30)$$

The normalization factor is formally,

$$N_{[t_1, t_2], \mu} = \int_{\substack{\phi|_{t_1}=0 \\ \phi|_{t_2}=0}} \mathcal{D}\phi e^{iS_{[t_1, t_2], \mu}(\phi)}. \quad (31)$$

Separating the free propagator in (30) yields,

$$Z_{[t_1, t_2], \mu}(\varphi_1, \varphi_2) = \frac{N_{[t_1, t_2], \mu}}{N_{[t_1, t_2], 0}} Z_{[t_1, t_2], 0}(\varphi_1, \varphi_2) \exp \left(i \int d^4x \mu(x) \phi_{\text{cl}}(x) \right). \quad (32)$$

Using the decomposition (12) we can rewrite this as,

$$Z_{[t_1, t_2], \mu}(\varphi_1, \varphi_2) = \frac{N_{[t_1, t_2], \mu}}{N_{[t_1, t_2], 0}} Z_{[t_1, t_2], 0}(\varphi_1, \varphi_2) \exp \left(\int d^3x (\mu_1(x) \varphi_1(x) + \mu_2(x) \varphi_2(x)) \right). \quad (33)$$

Here, μ_1 and μ_2 are defined as follows,

$$\mu_1(x) := i \int_{t_1}^{t_2} dt \frac{\sin \omega(t_2 - t)}{\sin \omega(t_2 - t_1)} \mu(t, x), \quad \mu_2(x) := i \int_{t_1}^{t_2} dt \frac{\sin \omega(t - t_1)}{\sin \omega(t_2 - t_1)} \mu(t, x). \quad (34)$$

Denoting the associated time-evolution operator by $U_{[t_1, t_2], \mu}$, the transition amplitude between coherent states is,

$$\begin{aligned} \langle \psi_{t_2, \eta_2} | U_{[t_1, t_2], \mu} | \psi_{t_1, \eta_1} \rangle &= K_{t_1, \eta_1} \overline{K_{t_2, \eta_2}} \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \\ &\exp \left(\int \frac{d^3x d^3k}{(2\pi)^3} \left(\eta_1(k) e^{-i(Et_1 - kx)} \varphi_1(x) + \overline{\eta_2(k)} e^{i(Et_2 - kx)} \varphi_2(x) \right) \right) \psi_0(\varphi_1) \overline{\psi_0(\varphi_2)} Z_{[t_1, t_2], \mu}(\varphi_1, \varphi_2). \end{aligned} \quad (35)$$

To evaluate this expression, we observe that we can combine the exponential factor appearing in (33) with the exponential in the above formula. We can then reduce the transition amplitude (up to normalization) to one of the free theory, but between modified coherent states defined by complex functions $\tilde{\eta}_1$ and $\tilde{\eta}_2$ given by

$$\tilde{\eta}_1(k) := \eta_1(k) + \int d^3x e^{i(Et_1 - kx)} \mu_1(x) \quad \text{and} \quad \tilde{\eta}_2(k) := \eta_2(k) + \int d^3x e^{i(Et_2 - kx)} \overline{\mu_2(x)}. \quad (36)$$

We obtain

$$\langle \psi_{t_2, \eta_2} | U_{[t_1, t_2], \mu} | \psi_{t_1, \eta_1} \rangle = \langle \psi_{\tilde{\eta}_2} | \psi_{\tilde{\eta}_1} \rangle \frac{N_{[t_1, t_2], \mu} K_{t_1, \eta_1} \overline{K_{t_2, \eta_2}}}{N_{[t_1, t_2], 0} K_{t_1, \tilde{\eta}_1} \overline{K_{t_2, \tilde{\eta}_2}}}. \quad (37)$$

Using (18) to express the inner product between the modified coherent states and (27) for the state normalization factors, equation (37) yields

$$\langle \psi_{t_2, \eta_2} | U_{[t_1, t_2], \mu} | \psi_{t_1, \eta_1} \rangle = \langle \psi_{\eta_2} | \psi_{\eta_1} \rangle \frac{N_{[t_1, t_2], \mu}}{N_{[t_1, t_2], 0}} \exp \left(i \int d^4x \mu(x) \hat{\eta}(x) \right) \exp \left(\frac{i}{2} \int d^4x \mu(x) \beta(x) \right), \quad (38)$$

where $\hat{\eta}$ is the complex classical solution of the Klein-Gordon equation determined by η_1 and η_2 via

$$\hat{\eta}(t, x) = \int \frac{d^3k}{(2\pi)^3 2E} \left(\eta_1(k) e^{-i(Et - kx)} + \overline{\eta_2(k)} e^{i(Et - kx)} \right). \quad (39)$$

On the other hand, any bounded solution of the Klein-Gordon equation can be expanded in this way. Hence, this establishes a one-to-one correspondence between pairs of coherent states parametrized by pairs of functions (η_1, η_2) and solutions $\hat{\eta}$.

The function $\beta(x)$ appearing in the last exponential of (38) is a solution of the Klein-Gordon equation whose Fourier decomposition is given by

$$\beta(t, k) = \int_{t_1}^{t_2} \frac{d\tau}{2E} \left(i e^{-iE(\tau - t)} + \frac{2 \sin(E(t - t_1)) \sin(E(t_2 - \tau))}{\sin(E(t_2 - t_1))} \right) \mu(\tau, k), \quad (40)$$

where

$$\beta(t, k) = 2E \int d^3x \beta(t, x) e^{-ikx}, \quad \text{and} \quad \mu(t, k) = 2E \int d^3x \mu(t, x) e^{-ikx}. \quad (41)$$

It remains to evaluate the propagator normalization factor (31) and combine the result with the other terms in (37). It is possible to relate this normalization factor $N_{[t_1, t_2], \mu}$ to that of the free theory $N_{[t_1, t_2], 0}$ defined in (7) by shifting the integration variable in (31) by a particular function α ,

$$\int_{\phi|_{t_1=0}}^{\phi|_{t_2=0}} \mathcal{D}\phi e^{iS_{[t_1, t_2], \mu}(\phi)} = \exp \left(\frac{i}{2} \int d^4x \mu(x) \alpha(x) \right) \int_{\phi|_{t_1=0}}^{\phi|_{t_2=0}} \mathcal{D}\phi e^{iS_{[t_1, t_2], 0}(\phi)}. \quad (42)$$

That is,

$$\frac{N_{[t_1, t_2], \mu}}{N_{[t_1, t_2], 0}} = \exp \left(\frac{i}{2} \int d^4x \mu(x) \alpha(x) \right), \quad (43)$$

where the function α is a solution of the inhomogeneous Klein-Gordon equation

$$(\square + m^2)\alpha(t, x) = \mu(t, x), \quad (44)$$

in the spacetime region $[t_1, t_2] \times \mathbb{R}^3$ and with the boundary conditions $\alpha(t_1, x) = 0$ and $\alpha(t_2, x) = 0$ for all $x \in \mathbb{R}^3$. It will be convenient to work in momentum space and consider the Fourier components of α ,

$$\alpha(t, k) = 2E \int d^3x \alpha(t, x) e^{-ikx}. \quad (45)$$

Equation (44) implies

$$(\partial_0^2 + E^2)\alpha(t, k) = \mu(t, k), \quad (46)$$

with $E^2 = k^2 + m^2$. The solution is easily found and can be written in the form

$$\alpha(t, k) = - \int_{t_1}^{t_2} \frac{d\tau}{2E} \left(\theta(t - \tau) 2 \sin(E(\tau - t)) + \frac{2 \sin(E(t - t_1)) \sin(E(t_2 - \tau))}{\sin(E(t_2 - t_1))} \right) \mu(\tau, k). \quad (47)$$

where $\theta(t)$ is the step function

$$\theta(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0. \end{cases} \quad (48)$$

Combining the exponential factor containing β in (38) with (43) amounts to summing the solution β given by (40) and α given by (47),

$$\gamma(t, k) := \alpha(t, k) + \beta(t, k) = \int_{t_1}^{t_2} \frac{d\tau}{2E} i \left(\theta(t - \tau) e^{iE(\tau - t)} + \theta(\tau - t) e^{-iE(\tau - t)} \right) \mu(\tau, k), \quad (49)$$

Since α solves the inhomogeneous equation (44) and β the homogeneous equation the sum γ solves the inhomogeneous equation (44) as well, but with different boundary conditions than α . Indeed, we can read off the boundary conditions from (49):

$$\begin{aligned} \text{for } t < t_1, \quad \gamma(t, k) &= \frac{e^{iEt}}{2E} \int_{t_1}^{t_2} d\tau i e^{-iE\tau} \mu(\tau, k), \\ \text{for } t > t_2, \quad \gamma(t, k) &= \frac{e^{-iEt}}{2E} \int_{t_1}^{t_2} d\tau i e^{iE\tau} \mu(\tau, k). \end{aligned} \quad (50)$$

Namely, the function γ contains only negative energy modes at early times ($t < t_1$) and positive energy modes at late times ($t > t_2$). We recognize these as the *Feynman boundary conditions*. Thus, we can write γ in the following integral form,

$$\gamma(x) = \int d^4x' G_F(x, x') \mu(x'), \quad (51)$$

where G_F is the Feynman propagator normalized such that $(\square_x + m^2)G_F(x, x') = \delta^4(x - x')$.

Now inserting (43) into (38) and using (49) as well as (51) we obtain for the transition amplitude,

$$\langle \psi_{t_2, \eta_2} | U_{[t_1, t_2], \mu} | \psi_{t_1, \eta_1} \rangle = \langle \psi_{\eta_2} | \psi_{\eta_1} \rangle \exp \left(i \int d^4x \mu(x) \hat{\eta}(x) \right) \exp \left(\frac{i}{2} \int d^4x d^4x' \mu(x) G_F(x, x') \mu(x') \right). \quad (52)$$

This expression is independent of the times t_1 and t_2 as long as the support of μ vanishes outside the interval $[t_1, t_2]$. Thus, we can take the limits $t_1 \rightarrow -\infty$ and $t_2 \rightarrow +\infty$, subsequently lift the restriction on the support of μ , and interpret the result as the S-matrix for the theory with the source interaction,

$$\langle \psi_{\eta_2} | S_\mu | \psi_{\eta_1} \rangle = \langle \psi_{\eta_2} | S_0 | \psi_{\eta_1} \rangle \exp \left(i \int d^4x \mu(x) \hat{\eta}(x) \right) \exp \left(\frac{i}{2} \int d^4x d^4x' \mu(x) G_F(x, x') \mu(x') \right). \quad (53)$$

C. General interactions

The result of the previous section can be combined with functional derivative techniques to work out the S-matrix in the case of a general interaction. The action of the scalar field with an arbitrary potential V in the spacetime region M can be written as

$$S_{M, V}(\phi) = S_{M, 0}(\phi) + \int_M d^4x V(x, \phi(x)). \quad (54)$$

We notice the usual functional identity,

$$\exp(iS_{M,V}(\phi)) = \exp\left(i \int_M d^4x V\left(x, -i \frac{\partial}{\partial \mu(x)}\right)\right) \exp(iS_{M,\mu}(\phi)) \Big|_{\mu=0}, \quad (55)$$

where $S_{M,\mu}$ is the action in the presence of a source interaction, defined in (29).

As above, at first the spacetime region of interest is determined by a time interval $[t_1, t_2]$ and we assume that the interaction vanishes outside of it,

$$V((t, x), \phi(t, x)) = 0, \forall x \in \mathbb{R}^3, \forall t \notin [t_1, t_2]. \quad (56)$$

Inserting (55) into the path integral of the propagator (5), we observe that we can move the exponential containing the functional derivative out of the integral to the front. In fact, we can repeat all steps identically as for the theory with source, but with the functional derivative term in front. Hence, the transition amplitude can be read off from (52) in combination with (55),

$$\langle \psi_{t_2,2} | U_{[t_1,t_2],V} | \psi_{t_1,1} \rangle = \exp\left(i \int d^4x V\left(x, -i \frac{\partial}{\partial \mu(x)}\right)\right) \langle \psi_{t_2,2} | U_{[t_1,t_2],\mu} | \psi_{t_1,1} \rangle \Big|_{\mu=0}. \quad (57)$$

In this expression a restriction to coherent states is not necessary, so we have written it for general states. We now recall that the transition amplitude (52) does not actually depend on the times t_1 and t_2 , but is identical to the S-matrix of the theory with source. Hence, expression (57) also does not depend on t_1 and t_2 and the limits $t_1 \rightarrow -\infty$ and $t_2 \rightarrow \infty$ are trivial. We can lift the restriction (56) on V and obtain the S-matrix,

$$\langle \psi_2 | S_V | \psi_1 \rangle = \exp\left(i \int d^4x V\left(x, -i \frac{\partial}{\partial \mu(x)}\right)\right) \langle \psi_2 | S_\mu | \psi_1 \rangle \Big|_{\mu=0}. \quad (58)$$

IV. GENERAL BOUNDARIES AND THE HYPERCYLINDER

The general boundary formulation of quantum theory posits that we can assign physically meaningful amplitudes not only to transitions between an initial and a final spacelike hypersurface, but to more general bounded regions of spacetime. Similarly, we can associate states (and state spaces) not only to spacelike hypersurfaces, but to more general hypersurfaces. A key conjecture in this context is that standard quantum field theory admits such a generalization – for a certain class of regions and hypersurfaces that are yet to be determined. In fact, the present paper may be viewed as proving that conjecture for a certain very limited class of regions and hypersurfaces. Nevertheless, this class is very different from the usual time-interval regions and equal-time hypersurfaces and hence points to more general geometries.

We follow the Schrödinger-Feynman approach to quantization above, adapting it to the more general context. Hence, the state space \mathcal{H}_Σ for a hypersurface Σ should be the space of functions on field configurations K_Σ on Σ . We write the inner product there (naively) as

$$\langle \psi_2 | \psi_1 \rangle = \int_{K_\Sigma} \mathcal{D}\varphi \psi_1(\varphi) \overline{\psi_2(\varphi)}. \quad (59)$$

The amplitude for a region M and a state ψ in the state space $\mathcal{H}_{\partial M}$ associated to the boundary ∂M of M is the integral

$$\rho_M(\psi) = \int_{K_\Sigma} \mathcal{D}\varphi \psi(\varphi) Z_M(\varphi), \quad (60)$$

where the hypersurface $\Sigma = \partial M$ represents the boundary of M . The quantity Z_M is the propagator given by the Feynman path integral,

$$Z_M(\varphi) = \int_{K_M, \phi|_\Sigma = \varphi} \mathcal{D}\phi e^{iS_M(\phi)}, \quad \forall \varphi \in K_\Sigma. \quad (61)$$

Here the integral is over the space K_M of field configurations ϕ in the interior of M such that ϕ agrees with φ on the boundary Σ .

When Σ is an equal-time hypersurface, the inner product (59) becomes (3). Similarly, when $M = [t_1, t_2] \times \mathbb{R}^3$, i.e., it is a time-interval times all of space in Minkowski spacetime, its boundary ∂M consists of the disjoint union of two equal-time hypersurfaces, Σ_1 defined by the time t_1 and Σ_2 defined by t_2 . Then, the amplitude (60) specializes to the transition amplitude (4) and the propagator (5) to (61). In what follows, we shall be interested in a very different type of geometry.

A. Klein-Gordon theory on the hypercylinder

1. Classical theory

Consider a sphere of radius R centered at the origin in space and extended to all of time. We refer to this hypersurface $\mathbb{R} \times S_R^2$ in Minkowski space as the *hypercylinder* of radius R , and to its interior $\mathbb{R} \times B_R^3$ as the *solid hypercylinder* of radius R . To describe fields on its boundary or in its interior it is convenient to use spherical coordinates in space, defined by three parameters: $\theta \in [0, \pi[$, $\phi \in [0, 2\pi[$ and $r \in [0, \infty[$. Bounded solutions of the Klein-Gordon equation may be expanded in products of spherical harmonics, spherical Bessel functions and exponentials. Consider a general expansion of the form

$$\phi(t, r, \Omega) = \int_{-\infty}^{\infty} dE \sum_{l=0}^{\infty} \sum_{m=-l}^l \alpha_{l,m}(E) e^{-iEt} f_l(E, r) Y_l^m(\Omega), \quad (62)$$

where f_l denotes a certain kind of spherical Bessel function to be discussed below. Ω is a collective notation for the angle coordinates (θ, ϕ) . Y_l^m denotes the spherical harmonic. See Appendix B for their definition and properties. The spherical harmonics satisfy the differential equation

$$(\Delta_{\Omega} Y_l^m)(\Omega) = -\frac{l(l+1)}{r^2} Y_l^m(\Omega), \quad \text{with} \quad \Delta_{\Omega} = \frac{\cos \theta}{r^2 \sin \theta} \partial_{\theta} + \frac{1}{r^2} \partial_{\theta}^2 + \frac{1}{r^2 \sin^2 \theta} \partial_{\phi}^2. \quad (63)$$

For (62) to solve the Klein-Gordon equation the function f_l must satisfy the differential equation,

$$(\Delta_r f_l)(E, r) = \left(-E^2 + m^2 + \frac{l(l+1)}{r^2} \right) f_l(E, r), \quad \text{with} \quad \Delta_r = \frac{2}{r} \partial_r + \partial_r^2. \quad (64)$$

If $E^2 \geq m^2$, this is solved by the *spherical Bessel functions* of the first kind j_l and of the second kind n_l , setting $f_l(E, r) = j_l(r\sqrt{E^2 - m^2})$ or $f_l(E, r) = n_l(r\sqrt{E^2 - m^2})$ respectively. Both, j_l and n_l are real. If $E^2 \leq m^2$, this is solved by the *modified spherical Bessel functions* of the first kind i_l^+ and of the second kind i_l^- , setting $f_l(E, r) = i_l^+(r\sqrt{m^2 - E^2})$ or $f_l(E, r) = i_l^-(r\sqrt{m^2 - E^2})$ respectively. For the definition of the different types of spherical Bessel functions and their relations, see Appendix B.

Not all types of Bessel functions lead to solutions of the Klein-Gordon equation that are well-defined and bounded everywhere:

- The ordinary and modified spherical Bessel functions of the second kind, n_l and i_l^- respectively are singular at the origin. So the associated solutions are singular on the time axis.
- The spherical Bessel functions of the first and second kind, j_l and n_l respectively decay with the inverse of the radius for large radius and so do the associated solutions.
- The modified spherical Bessel functions of the first and second kind, i_l^+ and i_l^- respectively grow exponentially for large radius and so do the associated solutions.

Hence, the only solutions that are well defined and bounded in all of spacetime arise from the spherical Bessel functions of the first kind and for $E^2 \geq m^2$.

It is sometimes useful to change the basis of the space of solutions of (64) and use ordinary or modified spherical Bessel functions of the third kind instead. Concretely, the spherical Bessel functions of the third kind are defined as,

$$h_l = j_l + in_l, \quad \text{and} \quad \bar{h}_l = j_l - in_l. \quad (65)$$

These are complex and have asymptotic behaviours for $z \rightarrow \infty$ given by

$$h_l(z) \rightarrow i^{-l-1} \frac{e^{iz}}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \quad \bar{h}_l(z) \rightarrow i^{l+1} \frac{e^{-iz}}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \quad (66)$$

which means that the corresponding solutions (62) are outgoing (h_l) and incoming (\bar{h}_l) waves. We also note that h_l and \bar{h}_l have no zeros on the positive real axis. The modified spherical Bessel functions of the third kind are the linear combinations $i_l^+ + i_l^-$ and $i_l^+ - i_l^-$. Note that the first of these shows exponential decay at large radius while the second one shows exponential growth. Of interest to us will be only the first one, which for $z \rightarrow \infty$ behaves as

$$i_l^+(z) + i i_l^-(z) \rightarrow i^{-l-2} e^{-z} \left(\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \right). \quad (67)$$

Note that this function is either real or imaginary (depending on l).

In order to allow a unified treatment of the cases $E^2 > m^2$ and $E^2 < m^2$, we make the following definitions,

$$a_l(E, r) := \begin{cases} j_l(r\sqrt{E^2 - m^2}) & \text{if } E^2 > m^2, \\ i_l^+(r\sqrt{m^2 - E^2}) & \text{if } E^2 < m^2, \end{cases} \quad \text{and} \quad b_l(E, r) := \begin{cases} n_l(r\sqrt{E^2 - m^2}) & \text{if } E^2 > m^2, \\ i_l^-(r\sqrt{m^2 - E^2}) & \text{if } E^2 < m^2, \end{cases} \quad (68)$$

$$\text{as well as} \quad c_l(E, r) := a_l(E, r) + i b_l(E, r), \quad \text{and} \quad p := \begin{cases} \sqrt{E^2 - m^2} & \text{if } E^2 > m^2, \\ i\sqrt{m^2 - E^2} & \text{if } E^2 < m^2. \end{cases} \quad (69)$$

2. Quantum theory

We now turn to the quantum theory. The first object of interest is the state space associated with a hypercylinder of radius R . Since we are dealing with the Schrödinger representation this should be the space of complex functions on the space of field configurations on $\mathbb{R} \times S_R^2$. Naively, one would allow essentially any real valued function as a configuration. However, it was shown in [6] that it makes sense to restrict to such configurations which extend to bounded classical solutions in all of space-time. These configurations were called *physical configurations*. Recalling the above discussion of solutions induced by different types of Bessel functions, this implies a restriction of the Fourier expansion of a configuration in the temporal direction to energy eigenvalues E such that $E^2 > m^2$. In terms of the particle spectrum it means that we restrict to the spectrum appearing also in the standard context of equal-time hypersurfaces where the square $p^2 = E^2 - m^2$ of the momentum is non-negative. While we shall keep this restriction for asymptotic states when considering the S-matrix, we will see that the interacting theory requires the consideration of more general configurations and corresponding states in intermediate calculations. We will thus extend the relevant structures of [6] here for that case.

Recall that there are two versions of each hypersurface – one for each possible orientation. In the case of the hypercylinder we can think of the orientation as choosing either the *inside* or the *outside* of the hypercylinder. To distinguish these, superscripts O (for outside) and I (for inside) were used in [6]. Here we shall only require the outside version of state spaces and hence do not write any explicit superscript. In particular, it is clear that the induced orientation of the hypercylinder as the boundary of the solid hypercylinder is the outside one. We call this state space \mathcal{H}_R . Hence the amplitude function ρ_R on a solid hypercylinder of radius R is evaluated with an outside state $\psi \in \mathcal{H}_R$,

$$\rho_R(\psi) = \int \mathcal{D}\varphi \psi(\varphi) Z_R(\varphi). \quad (70)$$

Here, Z_R is the propagator (61) for M the solid hypercylinder.

For the free Klein-Gordon theory we call this propagator $Z_{R,0}$ and its path integral expression can be evaluated in the same way as for the standard propagator (5). We shift the integration by a classical solution ϕ_{cl} matching the boundary configuration φ at radius R to get

$$Z_{R,0}(\varphi) = N_{R,0} e^{iS_{R,0}(\phi_{\text{cl}})}, \quad \text{with} \quad N_{R,0} = \int_{\phi|_{R=0}} \mathcal{D}\phi e^{iS_{R,0}(\phi)}. \quad (71)$$

Note that we can rewrite the action (1) on a classical solution as a boundary integral. Concretely, on the hypercylinder this is

$$S_{R,0}(\phi_{\text{cl}}) = -\frac{1}{2} \int dt d\Omega R^2 \phi_{\text{cl}}(t, R, \Omega) (\partial_r \phi_{\text{cl}})(t, R, \Omega). \quad (72)$$

The classical solution in question must be well defined inside the solid hypercylinder. This means it can be expanded in the form (62) with $f_l = a_l$, i.e., using ordinary and modified spherical Bessel functions of the first kind, which are regular everywhere. Since solutions in the interior are essentially in one-to-one correspondence to boundary configurations we can relate the two via

$$\phi_{\text{cl}}(t, r, \Omega) = \frac{a_l(E, r)}{a_l(E, R)} \varphi(t, \Omega). \quad (73)$$

This allows us to read off the propagator,

$$Z_{R,0}(\varphi) = N_{R,0} \exp \left(-\frac{1}{2} \int dt d\Omega \varphi(t, \Omega) iR^2 \frac{a'_l(E, R)}{a_l(E, R)} \varphi(t, \Omega) \right), \quad (74)$$

where a'_l is the derivative of a_l with respect to r and $R^2 \frac{a'_l(E, R)}{a_l(E, R)}$ is to be understood as an operator via the mode decomposition of the field. This generalizes the corresponding result of [6] to include non-physical configurations.¹

We will be interested also in a different propagator, namely the one associated with the region bounded by two hypercylinders of different radii, say R and $\hat{R} > R$. Again, we can evaluate the path integral by shifting with a classical solution. However, the class of solutions to be considered is now larger. In particular, we need not insist that a solution is well defined at the time axis, since the latter is not part of the region we consider. Concretely, we also need to admit ordinary and modified spherical Bessel functions of the second kind, meaning that in the expansion (62) we need not only admit $f_l = a_l$, but also $f_l = b_l$. Extending the result of [6] (and using slightly different conventions for δ and σ) the propagator is,

$$Z_{[R, \hat{R}], 0}(\varphi, \hat{\varphi}) = N_{[R, \hat{R}], 0} \exp \left(-\frac{1}{2} \int dt d\Omega (\varphi \quad \hat{\varphi}) W_{[R, \hat{R}]} \begin{pmatrix} \varphi \\ \hat{\varphi} \end{pmatrix} \right), \quad (75)$$

with

$$W_{[R, \hat{R}]} = \frac{i}{\delta_l(E, R, \hat{R})} \begin{pmatrix} R^2 \sigma_l(E, \hat{R}, R) & -\frac{1}{p} \\ -\frac{1}{p} & \hat{R}^2 \sigma_l(E, R, \hat{R}) \end{pmatrix}. \quad (76)$$

The function δ_l and σ_l are to be understood as operators defined as

$$\delta_l(E, R, \hat{R}) = a_l(E, R) b_l(E, \hat{R}) - b_l(E, R) a_l(E, \hat{R}), \quad (77)$$

$$\sigma_l(E, R, \hat{R}) = a_l(E, R) b'_l(E, \hat{R}) - b_l(E, R) a'_l(E, \hat{R}). \quad (78)$$

Again, the derivative refers to the second argument. Note that both δ_l and σ_l are always real for $E^2 > m^2$ and imaginary for $E^2 < m^2$.

We do not repeat here the calculation of [6] to obtain the vacuum wave function, but just import the result, extended to non-physical configurations by analytic continuation,²

$$\psi_{R, 0}(\varphi) = C_R \exp \left(-\frac{1}{2} \int dt d\Omega \varphi(t, \Omega) (B_R \varphi)(t, \Omega) \right). \quad (79)$$

Here C_R is a normalization factor and B_R denotes the family of operators indexed by the radius R given by

$$B_R = \frac{1 - ipR^2(a_l(E, R)a'_l(E, R) + b_l(E, R)b'_l(E, R))}{p(a_l^2(E, R) + b_l^2(E, R))} = -iR^2 \frac{c'_l(E, R)}{c_l(E, R)}. \quad (80)$$

Note that for $E^2 < m^2$ this operator becomes purely imaginary.

B. Coherent states on the hypercylinder

We use the following decomposition of field configurations on the hypercylinder,

$$\varphi(t, \Omega) = \int_{-\infty}^{\infty} dE \sum_{l, m} \varphi_{l, m}(E) e^{-iEt} Y_l^m(\Omega), \quad \varphi_{l, m}(E) = \frac{1}{2\pi} \int dt d\Omega \varphi(t, \Omega) e^{iEt} Y_l^{-m}(\Omega). \quad (81)$$

We define the family of states on the hypercylinder of radius R ,

$$\psi_{R, \eta}(\varphi) = K_{R, \eta} \exp \left(\int_{|E| \geq m} dE \sum_{l, m} \eta_{l, m}(E) \varphi_{l, m}(E) \right) \psi_{R, 0}(\varphi), \quad (82)$$

¹ We remark that the inclusion of non-physical configurations also modifies the normalization factor $N_{R, 0}$ as compared to [6], in spite of the innocent looking expression (71). However, the explicit form of this and other similar normalization factors is not of importance here. They can easily be computed using the same method as in [6].

² Note that the vacuum for $E^2 < m^2$ is determined here by analytically continuing the *outside* vacuum state. Instead analytically continuing the *inside* vacuum state would lead to a different vacuum. These two choices correspond to the ambiguity described for $E^2 > m^2$ in [6].

parametrized by the complex functions $\eta_{l,m}(E)$. Here, $K_{R,\eta}$ is a normalization factor that will be determined later and $\psi_{R,0}$ is the vacuum state (79). We show in the following that these states are coherent states in complete analogy to the usual coherent states on equal-time hypersurfaces: They form a complete set of states, remain coherent under “evolution” etc. (See also the end of Appendix A for a justification of the definition (82) in terms of creation operators.) The restriction of η to the range $E^2 > m^2$ means that we restrict to a dependence on physical configurations. Sometimes we consider $\eta_{l,m}$ evaluated on any real value E . Then we define $\eta_{l,m}(E) = 0$ if $E^2 < m^2$.

The inner product (59) of coherent states is

$$\langle \psi_{R,\eta'} | \psi_{R,\eta} \rangle = \overline{K_{R,\eta'}} K_{R,\eta} \int \mathcal{D}\varphi \exp \left(\int_{|E| \geq m} dE \sum_{l,m} \left(\overline{\eta'_{l,m}(E)} \overline{\varphi_{l,m}(E)} + \eta_{l,m}(E) \varphi_{l,m}(E) \right) \right) |\psi_{R,0}(\varphi)|^2. \quad (83)$$

To evaluate this we notice from (79) that,³

$$|\psi_{R,0}(\varphi)|^2 = |C_R|^2 \exp \left(- \int_{|E| \geq m} dE \sum_{l,m} \varphi_{l,m}(E) \frac{2\pi}{p|h_l(pR)|^2} \varphi_{l,-m}(-E) \right). \quad (84)$$

Since φ is real we have $\overline{\varphi_{l,m}(E)} = \varphi_{l,-m}(-E)$ and,

$$\begin{aligned} \langle \psi_{R,\eta'} | \psi_{R,\eta} \rangle = \overline{K_{R,\eta'}} K_{R,\eta} |C_R|^2 \int \mathcal{D}\varphi \exp \left(\int_{|E| \geq m} dE \sum_{l,m} \left(\left(\overline{\eta'_{l,-m}(-E)} + \eta_{l,m}(E) \right) \varphi_{l,m}(E) \right. \right. \\ \left. \left. - \varphi_{l,m}(E) \frac{2\pi}{p|h_l(pR)|^2} \varphi_{l,-m}(-E) \right) \right). \end{aligned} \quad (85)$$

The integral in φ can be evaluated with the usual technique of shifting the integration variable. In particular, we implement for $E^2 > m^2$ the shift

$$\varphi_{l,m}(E) \rightarrow \varphi_{l,m}(E) + \frac{p|h_l(pR)|^2}{4\pi} \left(\overline{\eta'_{l,m}(E)} + \eta_{l,-m}(-E) \right). \quad (86)$$

$\varphi_{l,m}(E)$ remains unshifted for $E^2 < m^2$. We obtain the following expression for the inner product,

$$\langle \psi_{R,\eta'} | \psi_{R,\eta} \rangle = \overline{K_{R,\eta'}} K_{R,\eta} \exp \left(\int_{|E| \geq m} dE \sum_{l,m} \frac{p|h_l(pR)|^2}{8\pi} \left(\overline{\eta'_{l,m}(E)} + \eta_{l,-m}(-E) \right) \left(\eta_{l,m}(E) + \overline{\eta'_{l,-m}(-E)} \right) \right). \quad (87)$$

We fix the factors $K_{R,\eta}$ such that the coherent states are normalized. As is easy to see, this is achieved by setting

$$K_{R,\eta} = \exp \left(- \int_{|E| \geq m} dE \sum_{l,m} \frac{p|h_l(pR)|^2}{8\pi} \left(\eta_{l,m}(E) \eta_{l,-m}(-E) + |\eta_{l,m}(E)|^2 \right) \right), \quad (88)$$

which yields,

$$\langle \psi_{R,\eta'} | \psi_{R,\eta} \rangle = \exp \left(\int_{|E| \geq m} dE \sum_{l,m} \frac{p|h_l(pR)|^2}{4\pi} \left(\eta_{l,m}(E) \overline{\eta'_{l,m}(E)} - \frac{1}{2} |\eta_{l,m}(E)|^2 - \frac{1}{2} |\eta'_{l,m}(E)|^2 \right) \right). \quad (89)$$

The coherent states satisfy the completeness relation

$$D^{-1} \int d\eta d\overline{\eta} |\psi_{R,\eta}\rangle \langle \psi_{R,\eta}| = I, \quad (90)$$

³ Note that the vacuum is defined on unphysical configurations as well, but these do not appear here explicitly because $B_R + \overline{B_R} = 0$ if $E^2 < m^2$.

where the constant D is given by

$$D = \int d\eta d\bar{\eta} \exp \left(- \int_{|E| \geq m} dE \sum_{l,m} \frac{p|h_l(pR)|^2}{4\pi} |\eta_{l,m}(E)|^2 \right). \quad (91)$$

We can show the correctness of the completeness relation (90) inserting it in an inner product between two coherent states,

$$\begin{aligned} \langle \psi_{R,\eta} | \psi_{R,\eta'} \rangle &= \langle \psi_{R,\eta} | D^{-1} \int d\eta'' d\bar{\eta}'' |\psi_{R,\eta''}\rangle \langle \psi_{R,\eta''} | \psi_{R,\eta'} \rangle \\ &= D^{-1} \exp \left(\int_{|E| \geq m} dE \sum_{l,m} \frac{p|h_l(pR)|^2}{4\pi} \left(-\frac{1}{2} |\eta_{l,m}(E)|^2 - \frac{1}{2} |\eta'_{l,m}(E)|^2 \right) \right) \times \\ &\quad \int d\eta'' d\bar{\eta}'' \exp \left(\int_{|E| \geq m} dE \sum_{l,m} \frac{p|h_l(pR)|^2}{4\pi} \left(\overline{\eta_{l,m}(E)} \eta''_{l,m}(E) + \overline{\eta'_{l,m}(E)} \eta'_{l,m}(E) - |\eta''_{l,m}(E)|^2 \right) \right). \end{aligned} \quad (92)$$

In order to calculate the integral we shift the integration variables by the quantities

$$\eta''_{l,m}(E) \rightarrow \eta''_{l,m}(E) + \eta'_{l,m}(E), \quad \text{and} \quad \overline{\eta''_{l,m}(E)} \rightarrow \overline{\eta''_{l,m}(E)} + \overline{\eta'_{l,m}(E)}. \quad (93)$$

So we obtain

$$\begin{aligned} \langle \psi_{R,\eta} | \psi_{R,\eta'} \rangle &= \exp \left(\int_{|E| \geq m} dE \sum_{l,m} \frac{p|h_l(pR)|^2}{4\pi} \left(\eta'_{l,m}(E) \overline{\eta_{l,m}(E)} - \frac{1}{2} |\eta_{l,m}(E)|^2 - \frac{1}{2} |\eta'_{l,m}(E)|^2 \right) \right) \times \\ &\quad D^{-1} \int d\eta'' d\bar{\eta}'' \exp \left(- \int_{|E| \geq m} dE \sum_{l,m} \frac{p|h_l(pR)|^2}{4\pi} |\eta''_{l,m}(E)|^2 \right). \end{aligned} \quad (94)$$

On the right-hand side of the above formula we recognize in the first line the expression of the inner product $\langle \psi_{R,\eta} | \psi_{R,\eta'} \rangle$, and using expression (91) the second line is equal to 1. This completes the proof of the completeness relation (90).

We also calculate the amplitude for a coherent state on a solid hypercylinder. Thus, we have to evaluate the integral

$$\rho_{R,0}(\psi_{R,\eta}) = \int \mathcal{D}\varphi \psi_{R,\eta}(\varphi) Z_{R,0}(\varphi), \quad (95)$$

where the propagator $Z_{R,0}$ is given by (74). This yields,

$$\rho_{R,0}(\psi_{R,\eta}) = K_{R,\eta} C_R N_{R,0} \int \mathcal{D}\varphi \exp \left(\int_{-\infty}^{\infty} dE \sum_{l,m} \left(\eta_{l,m}(E) \varphi_{l,m}(E) - \varphi_{l,m}(E) \frac{\pi}{p c_l(E, R) a_l(E, R)} \varphi_{l,-m}(-E) \right) \right). \quad (96)$$

We shift the integration variable so that the mixed term generated by the second term in the exponential cancels the term $\eta_{l,m}(E) \varphi_{l,m}(E)$. The shift only concerns physical configurations and is given by

$$\varphi_{l,m}(E) \rightarrow \varphi_{l,m}(E) + \frac{p h_l(pR) j_l(pR)}{2\pi} \eta_{l,-m}(-E). \quad (97)$$

This yields

$$\rho_{R,0}(\psi_{R,\eta}) = \exp \left(\int_{|E| \geq m} dE \sum_{l,m} \frac{p}{8\pi} (h_l^2(pR) \eta_{l,m}(E) \eta_{l,-m}(-E) - |h_l(pR)|^2 |\eta_{l,m}(E)|^2) \right). \quad (98)$$

The natural analogue of time evolution for the hypercylinder geometry is radial evolution in space. The evolution of a coherent state from a hypercylinder of radius \hat{R} to a hypercylinder of radius $R < \hat{R}$ is determined by the propagator (75) via

$$\psi_{R,\eta}(\varphi) = \int \mathcal{D}\hat{\varphi} \psi_{\hat{R},\hat{\eta}}(\hat{\varphi}) Z_{[R,\hat{R}],0}(\varphi, \hat{\varphi}). \quad (99)$$

(The equation for $R > \hat{R}$ is analogous, but with inside states.) As is to be expected, a coherent state remains a coherent state under radial evolution. Indeed, the relation between η at R and $\hat{\eta}$ at \hat{R} turns out to be given by the equation

$$\eta_{l,m}(E) = \hat{\eta}_{l,m}(E) \frac{h_l(p\hat{R})}{h_l(pR)}. \quad (100)$$

The polynomial representation (discussed in Appendix A mainly for the equal-time hyperplane case) allows to expand coherent states in terms of multi-particle states,

$$\begin{aligned} \psi_{R,\eta} = \exp \left(- \int_{|E| \geq m} dE \sum_{l,m} \frac{p|h_l(pR)|^2}{8\pi} |\eta_{l,m}(E)|^2 \right) \\ \sum_{n=0}^{\infty} \frac{1}{n!} \int_{|E_1| \geq m} dE_1 \sum_{l_1,m_1} \cdots \int_{|E_n| \geq m} dE_n \sum_{l_n,m_n} \eta_{l_1,m_1}(E_1) \cdots \eta_{l_n,m_n}(E_n) \psi_{R,(E_1,l_1,m_1),\dots,(E_n,l_n,m_n)} \end{aligned} \quad (101)$$

Here, $\psi_{R,(E_1,l_1,m_1),\dots,(E_n,l_n,m_n)}$ denotes the state with n particles with the given quantum numbers. Note that our conventions for multi-particle states are different here from those of [6]. For simplicity, we take here the sign of the energy not to be a separate quantum number and also use a different normalization. The normalization is fixed by writing down the one-particle wave function which is,

$$\psi_{R,E,l,m}(\varphi) = \varphi_{l,m}(E) \psi_{R,0}. \quad (102)$$

We also note the inner product between a coherent state and a multi-particle state,

$$\begin{aligned} \langle \psi_{R,(E_1,l_1,m_1),\dots,(E_n,l_n,m_n)} | \psi_{R,\eta} \rangle = \exp \left(- \int_{|E| \geq m} dE \sum_{l,m} \frac{p|h_l(pR)|^2}{8\pi} |\eta_{l,m}(E)|^2 \right) \\ \eta_{l_1,m_1}(E_1) \cdots \eta_{l_n,m_n}(E_n) \frac{p_1|h_{l_1}(p_1R)|^2}{4\pi} \cdots \frac{p_n|h_{l_n}(p_nR)|^2}{4\pi}. \end{aligned} \quad (103)$$

V. ASYMPTOTIC AMPLITUDE ON THE HYPERCYLINDER

Suppose we are interested in a scattering process where interactions are not negligible at any time, such as a stationary process. A basic assumption underlying the standard treatment of scattering processes via the S-matrix as presented in Section III is then violated. However, suppose that at the same time interactions can be neglected as we go far away in space from the interaction center. Recalling Section IV A it is obvious that the hypercylinder geometry is well suited to describe this situation. Concretely, we consider amplitudes for free states on the hypercylinder of given radius R with interactions switched on inside, i.e., for radius $r < R$. We then take the asymptotic limit of this amplitude for $R \rightarrow \infty$.

In this section we show that this asymptotic amplitude exists and compute it in three different cases. First we consider the free Klein-Gordon theory, then we add a source field and finally we consider general interactions via functional methods. The derivation in this section proceeds substantially in parallel to the one of the standard S-matrix in Section III. We use the coherent states defined in Section IV B.

A. Free theory

The first step we need to take in order to make sense of asymptotic amplitudes is to switch to the interaction picture. Recall that the analogue of time evolution in the conventional picture is now radial evolution in space. Hence, the interaction picture means that we identify states on different hypercylinders if they are related by radial evolution in the free theory. For coherent states this radial evolution is given by equation (100). Since the product $\xi_{l,m}(E) := h_l(pR)\eta_{l,m}(E)$ is preserved under radial evolution a good way to define interaction picture coherent states on the hypercylinder is by inserting this relation into (82). We get the wave functions,

$$\psi_{R,\xi}(\varphi) = K_{R,\xi} \exp \left(\int_{|E| \geq m} dE \sum_{l,m} \frac{\xi_{l,m}(E)}{h_l(pR)} \varphi_{l,m}(E) \right) \psi_{R,0}(\varphi), \quad (104)$$

depending on complex functions $\xi_{l,m}(E)$. The normalization factor can be calculated from (88),

$$K_{R,\xi} = \exp \left(- \int_{|E| \geq m} dE \sum_{l,m} \frac{p}{8\pi} \left(\frac{\bar{h}_l(pR)}{h_l(pR)} \xi_{l,m}(E) \xi_{l,-m}(-E) + |\xi_{l,m}(E)|^2 \right) \right). \quad (105)$$

The amplitude of the interaction picture coherent state for the solid hypercylinder is obtained from (98),

$$\rho_{R,0}(\psi_{R,\xi}) = \exp \left(\int_{|E| \geq m} dE \sum_{l,m} \frac{p}{8\pi} (\xi_{l,m}(E) \xi_{l,-m}(-E) - |\xi_{l,m}(E)|^2) \right). \quad (106)$$

By construction this expression is independent of the radius R . Taking the limit $R \rightarrow \infty$ is hence trivial and we can write down the asymptotic amplitude $\mathcal{S}_0 : \mathcal{H}_\infty \rightarrow \mathbb{C}$ of the free theory,

$$\mathcal{S}_0(\psi_\xi) = \lim_{R \rightarrow \infty} \rho_{R,0}(\psi_{R,\xi}) = \exp \left(\int_{|E| \geq m} dE \sum_{l,m} \frac{p}{8\pi} (\xi_{l,m}(E) \xi_{l,-m}(-E) - |\xi_{l,m}(E)|^2) \right). \quad (107)$$

B. Theory with source

We now turn to the Klein-Gordon theory with an additional source field μ , given by the action (29). The spacetime region of interest is now the solid hypercylinder of radius R . We assume that the source field vanishes outside the solid hypercylinder, i.e., for radius $r \geq R$.

The path integral (61) determining the relevant field propagator $Z_{R,\mu}(\varphi)$ is evaluated as usual by shifting the integration variable by a solution ϕ_{cl} that matches the boundary data. ϕ_{cl} is defined inside the hypercylinder, i.e., for radius $r \leq R$. It equals φ at radius R which we write as $\phi_{\text{cl}}|_R = \varphi$. The propagator is then

$$Z_{R,\mu}(\varphi) = N_{R,\mu} e^{iS_{R,\mu}(\phi_{\text{cl}})}. \quad (108)$$

The normalization factor is given by

$$N_{R,\mu} = \int_{\phi|_R=0} \mathcal{D}\phi e^{iS_{R,\mu}(\phi)}. \quad (109)$$

Recall that ϕ_{cl} can be expanded in the form (62) with $f_l = a_l$ and the relation (73) between solutions and boundary configurations. This allows us to rewrite the source term of the action (29) evaluated on the classical solution as

$$\int_{\mathbb{R} \times B_R^3} d^4x \mu(x) \phi_{\text{cl}}(x) = \int_{-\infty}^{\infty} dE \sum_{l,m} \varphi_{l,m}(E) \frac{2\pi}{a_l(E, R)} M_{l,-m}(-E), \quad (110)$$

where the quantity $M_{l,m}(E)$ is defined as

$$M_{l,m}(E) := \int_0^\infty dr r^2 a_l(E, r) \mu_{l,m}(E, r). \quad (111)$$

$\mu_{l,m}(E, r)$ are the modes of the source field μ in the basis of the spherical harmonics via the expansion

$$\mu(t, r, \Omega) = \int_{-\infty}^{\infty} dE \sum_{l,m} \mu_{l,m}(E, r) e^{-iEt} Y_l^m(\Omega). \quad (112)$$

Then the propagator takes the form

$$Z_{R,\mu}(\varphi) = \frac{N_{R,\mu}}{N_{R,0}} Z_{R,0}(\varphi) \exp \left(\int_{-\infty}^{\infty} dE \sum_{l,m} \varphi_{l,m}(E) \frac{2\pi i}{a_l(E, R)} M_{l,-m}(-E) \right). \quad (113)$$

This expression is the analogue of (33). The amplitude (60) of the theory with source for a coherent state (104) on the solid hypercylinder is thus,

$$\rho_{R,\mu}(\psi_{R,\xi}) = \frac{N_{R,\mu}}{N_{R,0}} \int \mathcal{D}\varphi \psi_{R,\xi}(\varphi) \exp \left(\int_{-\infty}^{\infty} dE \sum_{l,m} \varphi_{l,m}(E) \frac{2\pi i}{a_l(E,R)} M_{l,-m}(-E) \right) Z_{R,0}(\varphi). \quad (114)$$

We write the coherent state wave function and the propagator explicitly to get,

$$\begin{aligned} \rho_{R,\mu}(\psi_{R,\xi}) = K_{R,\xi} C_R N_{R,\mu} \int \mathcal{D}\varphi \exp \left(\int_{-\infty}^{\infty} dE \sum_{l,m} \left(\frac{\xi_{l,m}(E)}{h_l(pR)} \varphi_{l,m}(E) \right. \right. \\ \left. \left. + \varphi_{l,m}(E) \frac{2\pi i}{a_l(E,R)} M_{l,-m}(-E) - \varphi_{l,m}(E) \frac{\pi}{p c_l(E,R) a_l(E,R)} \varphi_{l,-m}(-E) \right) \right). \end{aligned} \quad (115)$$

We eliminate the cross term between φ and M , but only for unphysical configurations with $E^2 < m^2$ via the shift,

$$\varphi_{l,m}(E) \rightarrow \varphi_{l,m}(E) + i p c_l(E,R) M_{l,m}(E). \quad (116)$$

Note that this shift is real. That is, the shifted field remains real (in position space). The corresponding shift for physical configurations would be complex. We arrive at,

$$\begin{aligned} \rho_{R,\mu}(\psi_{R,\xi}) = K_{R,\xi} \frac{N_{R,\mu}}{N_{R,0}} \int \mathcal{D}\varphi \exp \left(\int_{|E| \geq m} dE \sum_{l,m} \left(\frac{\xi_{l,m}(E)}{h_l(pR)} \varphi_{l,m}(E) + \varphi_{l,m}(E) \frac{2\pi i}{j_l(pR)} M_{l,-m}(-E) \right) \right. \\ \left. - \int_{-m}^m dE \sum_{l,m} M_{l,m}(E) \frac{\pi p c_l(E,R)}{a_l(E,R)} M_{l,-m}(-E) \right) \psi_{R,0}(\varphi) Z_{R,0}(\varphi). \end{aligned} \quad (117)$$

In order to deal with the remaining cross term between φ and M for physical configurations, we shift the coherent state. That is, we define a new coherent state via,

$$\tilde{\xi}_{l,m}(E) := \xi_{l,m}(E) + 2\pi i \frac{h_l(pR)}{j_l(pR)} M_{l,-m}(-E). \quad (118)$$

This yields,

$$\rho_{R,\mu}(\psi_{R,\xi}) = \frac{K_{R,\xi}}{K_{R,\tilde{\xi}}} \frac{N_{R,\mu}}{N_{R,0}} \rho_{R,0}(\psi_{R,\tilde{\xi}}) \exp \left(- \int_{-m}^m dE \sum_{l,m} M_{l,m}(E) \frac{\pi p c_l(E,R)}{a_l(E,R)} M_{l,-m}(-E) \right) \quad (119)$$

Substituting the expression of the free transition amplitude (106) and the normalization factor (105) for the shifted coherent state $\psi_{R,\tilde{\xi}}$, we arrive at

$$\begin{aligned} \rho_{R,\mu}(\psi_{R,\xi}) = K_{R,\xi} \frac{N_{R,\mu}}{N_{R,0}} \exp \left(\int_{-\infty}^{\infty} dE \sum_{l,m} \left(-M_{l,m}(E) \frac{\pi p c_l(E,R)}{a_l(E,R)} M_{l,-m}(-E) \right. \right. \\ \left. \left. + i p \xi_{l,m}(E) M_{l,m}(E) + \xi_{l,m}(E) \frac{p a_l(E,R)}{4\pi c_l(E,R)} \xi_{l,-m}(-E) \right) \right). \end{aligned} \quad (120)$$

Note that the term quadratic in M arises as a combination of the unphysical part in (119) and a corresponding physical part coming from the shifted coherent state.

Substituting the expression of the normalization factor $K_{R,\xi}$ given in (105) we obtain

$$\begin{aligned} \rho_{R,\mu}(\psi_{R,\xi}) = \frac{N_{R,\mu}}{N_{R,0}} \exp \left(\int_{-\infty}^{\infty} dE \sum_{l,m} \left[\frac{p}{8\pi} (\xi_{l,m}(E) \xi_{l,-m}(-E) - |\xi_{l,m}(E)|^2) \right. \right. \\ \left. \left. - M_{l,m}(E) \frac{\pi p c_l(E,R)}{a_l(E,R)} M_{l,-m}(-E) + i p \xi_{l,m}(E) M_{l,m}(E) \right] \right). \end{aligned} \quad (121)$$

We recognize in the first line the amplitude of the coherent state in the case of the free theory (106), so

$$\rho_{R,\mu}(\psi_{R,\xi}) = \rho_{R,0}(\psi_{R,\xi}) \frac{N_{R,\mu}}{N_{R,0}} \exp \left(\int_{-\infty}^{\infty} dE \sum_{l,m} \left[-M_{l,m}(E) \frac{\pi p c_l(E, R)}{a_l(E, R)} M_{l,-m}(-E) + i p \xi_{l,m}(E) M_{l,m}(E) \right] \right). \quad (122)$$

The second term in the exponential can be rewritten as follows,

$$\int_{|E| \geq m} dE \sum_{l,m} p \xi_{l,m}(E) M_{l,m}(E) = \int d^4x \mu(x) \hat{\xi}(x), \quad (123)$$

with $\hat{\xi}$ given by

$$\hat{\xi}(t, r, \Omega) := \int_{|E| \geq m} dE \sum_{l,m} \frac{p}{2\pi} \xi_{l,m}(E) j_l(pr) e^{iEt} Y_l^{-m}(\Omega). \quad (124)$$

This implies that we have a one-to-one correspondence between coherent states parametrized by functions ξ and complex solutions $\hat{\xi}$ of the Klein-Gordon equation. Note that the restriction to physical configurations with $E^2 < m^2$ precisely corresponds to restricting solutions to be globally bounded.

We rewrite the first term in the exponential appearing in (122) as,

$$\exp \left(- \int_{-\infty}^{\infty} dE \sum_{l,m} M_{l,m}(E) \frac{\pi p c_l(E, R)}{a_l(E, R)} M_{l,-m}(-E) \right) = \exp \left(\frac{i}{2} \int d^4x \mu(x) \beta(x) \right), \quad (125)$$

with β a solution of the Klein-Gordon equation given by,

$$\beta_{l,m}(E, r) = i p a_l(E, r) \frac{c_l(E, R)}{a_l(E, R)} M_{l,m}(E) \quad (126)$$

where we expand,

$$\beta(t, r, \Omega) = \int_{-\infty}^{\infty} dE \sum_{l,m} \beta_{l,m}(E, r) e^{-iEt} Y_l^m(\Omega). \quad (127)$$

In analogy to the case for equal-time hyperplanes in equation (42) the normalization factor $N_{R,\mu}$ can be related to the normalization factor $N_{R,0}$. Explicitly,

$$\int_{\phi|_R=0} \mathcal{D}\phi e^{iS_{R,\mu}(\phi)} = \exp \left(\frac{i}{2} \int d^4x \mu(x) \alpha(x) \right) \int_{\phi|_R=0} \mathcal{D}\phi e^{iS_{R,0}(\phi)}, \quad (128)$$

which implies,

$$\frac{N_{R,\mu}}{N_{R,0}} = \exp \left(\frac{i}{2} \int d^4x \mu(x) \alpha(x) \right), \quad (129)$$

where α satisfies the inhomogeneous Klein-Gordon equation with vanishing boundary conditions at radius R ,

$$(\square + m^2)\alpha = \mu, \quad \text{and} \quad \alpha|_R = 0. \quad (130)$$

It will be convenient to express the exponential factor appearing in (129) in momentum space,

$$\exp \left(\frac{i}{2} \int d^4x \mu(x) \alpha(x) \right) = \exp \left(i\pi \int_{-\infty}^{\infty} dE \sum_{l,m} \int_0^{\infty} dr r^2 \mu_{l,-m}(-E, r) \alpha_{l,m}(E, r) \right), \quad (131)$$

where $\alpha_{l,m}(E, r)$ are the modes in the expansion

$$\alpha(t, r, \Omega) = \int_{-\infty}^{\infty} dE \sum_{l,m} \alpha_{l,m}(E, r) e^{-iEt} Y_l^m(\Omega). \quad (132)$$

The inhomogeneous Klein-Gordon equation satisfied by α induces a differential equation for the modes $\alpha_{l,m}(E, r)$. The solution results to be equal to

$$\alpha_{l,m}(E, r) = p a_l(E, r) \left(N_{l,m}(E, r) - N_{l,m}(E) + \frac{b_l(E, R)}{a_l(E, R)} M_{l,m}(E) \right) - p b_l(E, r) M_{l,m}(E, r), \quad (133)$$

where in addition to (111) we define,

$$N_{l,m}(E, r) := \int_0^r ds s^2 b_l(E, s) \mu_{l,m}(E, s), \quad (134)$$

$$M_{l,m}(E, r) := \int_0^r ds s^2 a_l(E, s) \mu_{l,m}(E, s), \quad (135)$$

$$N_{l,m}(E) := \int_0^\infty ds s^2 b_l(E, s) \mu_{l,m}(E, s). \quad (136)$$

Notice that for $r > R$, $N_{l,m}(E, r) = N_{l,m}(E)$ and $M_{l,m}(E, r) = M_{l,m}(E)$, because the source components $\mu_{l,m}$ vanish outside the radius R .

Combining the factors (125) and (129) means that we have to sum α and β , resulting in

$$\gamma_{l,m}(E, r) := \alpha_{l,m}(E, r) + \beta_{l,m}(E, r) = p a_l(E, r) (N_{l,m}(E, r) - N_{l,m}(E) + i M_{l,m}(E)) - p b_l(E, r) M_{l,m}(E, r). \quad (137)$$

Since β is a solution of the homogeneous equation, γ satisfies the same inhomogeneous equation as α , but with different boundary conditions. In particular, for $r > R$, γ becomes

$$\gamma_{l,m}(E, r) \Big|_{r>R} = i p M_{l,m}(E) c_l(E, r). \quad (138)$$

Now, recall from Section IV A that for $E^2 > m^2$ the function c_l becomes h_l and generates outgoing waves. For $E^2 < m^2$ the function $c_l(E, r)$ becomes $i_l^+ + i_l^-$ and generates solutions exponentially decaying with the radius. Hence, these are the boundary conditions that γ satisfies and that in fact determine it uniquely. This is to be compared with the γ obtained in (49) which is determined by the Feynman boundary conditions (50).

It will be useful to express γ in a different way using the definitions (111), (134), (135), (136), and (48),

$$\gamma_{l,m}(E, r) = i p \int_0^\infty ds s^2 \mu_{l,m}(E, s) \{ \theta(r-s) c_l(E, r) a_l(E, s) + \theta(s-r) c_l(E, s) a_l(E, r) \}. \quad (139)$$

In position space this is

$$\gamma(t, r, \Omega) = i \int_{-\infty}^\infty dE \sum_{l,m} \int_0^\infty ds s^2 p \mu_{l,m}(E, s) e^{-iEt} Y_l^m(\Omega) \{ \theta(r-s) c_l(E, r) a_l(E, s) + \theta(s-r) c_l(E, s) a_l(E, r) \}. \quad (140)$$

Substituting the inverse of the transformation (112) we get,

$$\begin{aligned} \gamma(t, r, \Omega) = & \frac{i}{2\pi} \int_{-\infty}^\infty dE \sum_{l,m} \int_0^\infty ds s^2 p \int dt' d\Omega' e^{iEt'} \overline{Y_l^m(\Omega')} \mu(t', s, \Omega') e^{-iEt} Y_l^m(\Omega) \times \\ & \times \{ \theta(r-s) c_l(E, r) a_l(E, s) + \theta(s-r) c_l(E, s) a_l(E, r) \}. \end{aligned} \quad (141)$$

The sum over l, m can be performed using the formula (B10) of Appendix B.

In the terms of the associated Green function,

$$\gamma(x) = \int d^4 x' G(x, x') \mu(x'), \quad (142)$$

where

$$G(t, x, t', x') = \int_{-\infty}^\infty dE \frac{1}{8\pi^2} \frac{e^{ip|x-x'| + iE(t-t')}}{|x-x'|}. \quad (143)$$

This Green function can be written in a more familiar form

$$\int_{-\infty}^{\infty} dE \frac{1}{8\pi^2} \frac{e^{ip|\underline{x}-\underline{x}'|+iE(t-t')}}{|\underline{x}-\underline{x}'|} = -\frac{1}{(2\pi)^4} \int \frac{e^{-ik(x-x')}}{k^2 - m^2 + i\epsilon} d^4k. \quad (144)$$

The right-hand side is the standard integral representation of the Feynman propagator of the massive scalar field. Hence $G = G_F$ and the γ of this section is identical to the γ of Section III B. This implies the surprising fact that for a bounded source the spatially asymptotic boundary conditions (138) are *equivalent* to the usual temporally asymptotic Feynman boundary conditions (50). Hence we can write the product of γ and μ arising in the combination of (125) and (129) as

$$\frac{i}{2} \int d^4x \mu(x) \gamma(x) = \frac{i}{2} \int d^4x d^4x' \mu(x) G_F(x, x') \mu(x'). \quad (145)$$

We insert (123), (125) and (129) into (122) and use (145) to obtain for the amplitude,

$$\rho_{R,\mu}(\psi_{R,\xi}) = \rho_R(\psi_{R,\xi}) \exp \left(i \int d^4x \hat{\xi}(x) \mu(x) \right) \exp \left(\frac{i}{2} \int d^4x d^4x' \mu(x) G_F(x, x') \mu(x') \right). \quad (146)$$

We notice that in this expression no explicit dependence on the radius R is present, as long as the support of the source field μ lies completely within this radius. So we can take the limit $R \rightarrow \infty$, lift the restriction on the support of μ , and interpret the result as the asymptotic amplitude in the case of a source interaction,

$$\mathcal{S}_\mu(\psi_\xi) = \mathcal{S}_0(\psi_\xi) \exp \left(i \int d^4x \hat{\xi}(x) \mu(x) \right) \exp \left(\frac{i}{2} \int d^4x d^4x' \mu(x) G_F(x, x') \mu(x') \right). \quad (147)$$

This is to be compared to (53).

C. General interactions

The transition amplitude in the case of a general interaction (54) can be computed applying the same functional derivative techniques used in Section III C. However, the region where we initially assume the interaction to vanish is different, i.e.,

$$V((t, x), \phi(t, x)) = 0, \quad \text{if } |x| \geq R. \quad (148)$$

Inserting (55) into the path integral of the propagator (61) where the region M is the solid hypercylinder of radius R , we observe that we can move the exponential containing the functional derivative out of the integral to the front. We can repeat all steps identically as for the theory with source, but with the functional derivative term in front. Hence, the amplitude can be read off from (146) in combination with (55),

$$\rho_{R,V}(\psi_R) = \exp \left(i \int d^4x V \left(x, -i \frac{\partial}{\partial \mu(x)} \right) \right) \rho_{R,\mu}(\psi_R) \Big|_{\mu=0}. \quad (149)$$

In this expression a restriction to coherent states is not necessary, so we have written it for general states. We now recall that the transition amplitude (146) does not actually depend on the radius R , but is identical to the asymptotic amplitude of the theory with source. Hence, expression (149) also does not depend on R and the limit $R \rightarrow \infty$ is trivial. We can lift the restriction (148) on V and obtain the asymptotic amplitude,

$$\mathcal{S}_V(\psi) = \exp \left(i \int d^4x V \left(x, -i \frac{\partial}{\partial \mu(x)} \right) \right) \mathcal{S}_\mu(\psi) \Big|_{\mu=0}. \quad (150)$$

This is to be compared to (58).

VI. EQUALITY OF ASYMPTOTIC AMPLITUDES

Although the hypercylinder geometry we used in Section V is rather different from the usual equal-time hyperplanes, we obtain asymptotic amplitudes that look very similar to the standard S-matrix expressions. While the similarity

between the expressions (58) and (150) is unsurprising, that of the underlying asymptotic amplitudes with source, (53) and (147), is striking. For example, the same (Feynman) propagator appears in both expressions. As we have seen this is due to a non-trivial equivalence between boundary conditions of the inhomogeneous Klein-Gordon equation. Namely, the Feynman boundary condition (50) at past and future temporal infinity is equivalent to an outgoing wave boundary condition (138) at spatial infinity.

From a physical point of view the similarity should not surprise us. If we want to describe a process that is bounded both in space and time (i.e., such that interactions may be neglected at large spatial and temporal distance) the hypercylinder geometry should serve as well as the standard one. Indeed, we show in this section that the asymptotic amplitudes in the two settings are *identical*, if the correct identification of temporal and spatial asymptotic states is performed.

In the standard setting, let us call the asymptotic state space in the infinite past \mathcal{H}_1 . Correspondingly, we call the asymptotic state space in the infinite future \mathcal{H}_2 . The tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ is the total Hilbert space of states at spatial infinity. (The dualization of the future state space occurs because initial states are ket-states while final ones are bra-states.) The S-matrix (58) is then a linear map from the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ to the complex numbers. In the hypercylinder setting, we call the asymptotic state space at infinite radius \mathcal{H}_{cyl} .

Identifying asymptotic states at temporal and at spatial infinity now translates to an isomorphism of Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2^* \rightarrow \mathcal{H}_{\text{cyl}}$. We can read off this isomorphism by comparing (53) and (147) as follows. Recall that we have a one-to-one correspondence between coherent states $|\psi_{\eta_1}\rangle \otimes |\psi_{\eta_2}\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2^*$ and complex classical solutions $\hat{\eta}$ in spacetime via (39). Similarly, we have a one-to-one correspondence between coherent states $\psi_\xi \in \mathcal{H}_{\text{cyl}}$ and complex classical solutions $\hat{\xi}$ in spacetime via (124). For the spatially asymptotic amplitude (147) to agree with the S-matrix (53) we obviously need to identify the classical solutions, i.e., $\hat{\xi} = \hat{\eta}$. Otherwise the μ -dependence would be different in the two expressions. It remains to check that this identification also makes the free amplitudes (28) and (107) equal.

Consider a bounded complex solution $\hat{\xi}$ of the Klein-Gordon equation in Minkowski spacetime. This solution corresponds via its decomposition into positive and negative frequency components (39) which we shall write also as $\hat{\xi} = \hat{\xi}_+ + \hat{\xi}_-$ to a pair of coherent states whose free S-matrix (28), i.e. inner product (18) is

$$\begin{aligned} \langle \psi_{\hat{\xi}_-} | \mathcal{S}_0 | \psi_{\hat{\xi}_+} \rangle &= \exp \left(\int \frac{d^3 k}{(2\pi)^{3/2} E} \left(\xi_+(k) \xi_-(k) - \frac{1}{2} |\xi_+(k)|^2 - \frac{1}{2} |\xi_-(k)|^2 \right) \right) \\ &= \exp \left(\int d^3 x \left(2\hat{\xi}_+(t, x)(\omega \hat{\xi}_-)(t, x) - \hat{\xi}_+(t, x)(\omega \hat{\xi}_+)(t, x) - \hat{\xi}_-(t, x)(\omega \hat{\xi}_-)(t, x) \right) \right) \end{aligned} \quad (151)$$

In the last line the time t is arbitrary. Inserting the decomposition of $\hat{\xi}_\pm$ in terms of spherical harmonics and Bessel functions,

$$\hat{\xi}_\pm(t, r, \Omega) = \int_m^\infty dE \sum_{l, m} \frac{p}{2\pi} j_l(pr) Y_l^{-m}(\Omega) \xi_{l, m}(\mp E) e^{\mp iEt}, \quad (152)$$

yields after integration over x ,

$$\langle \psi_{\hat{\xi}_-} | \mathcal{S}_0 | \psi_{\hat{\xi}_+} \rangle = \int_{|E| \geq m} dE \sum_{l, m} \frac{p}{8\pi} (\xi_{l, m}(E) \xi_{l, -m}(-E) - |\xi_{l, m}(E)|^2) = \mathcal{S}_0(\psi_\xi), \quad (153)$$

the free S-matrix on the hypercylinder (107), as required.

We turn now to look at the isomorphism $\mathcal{H}_1 \otimes \mathcal{H}_2^* \rightarrow \mathcal{H}_{\text{cyl}}$ in terms of multi-particle states. To this end we use the completeness relation for coherent states together with the formulas from Sections II B and IV B relating coherent states to multi-particle states. However, we have to transform these formulas first into the interaction picture. For the standard setting of equal-time hyperplanes this transformation is trivial and relations (19) as well as (22) and (23) simply remain the same in the interaction picture. In the hypercylinder setting the situation is different. Indeed, we first have to define particle states in the interaction picture. A suitable definition which we shall use in this section is,

$$\psi_{R, E, l, m}(\varphi) = \frac{\varphi_{l, m}(E)}{h_l(pR)} \psi_{R, 0}(\varphi), \quad (154)$$

for a one-particle wave function. This is to be contrasted with (102). The definition of multi-particle wave functions is then fixed in the usual way and the inner product between two n -particle states is

$$\langle \psi_{(l_1, m_1, E_1), \dots, (l_n, m_n, E_n)} | \psi_{(l'_1, m'_1, E'_1), \dots, (l'_n, m'_n, E'_n)} \rangle = \frac{p_1}{4\pi} \dots \frac{p_n}{4\pi} \delta_{l_1, l'_1} \dots \delta_{l_n, l'_n} \delta_{m_1, m'_1} \dots \delta_{m_n, m'_n} \delta(E_1 - E'_1) \dots \delta(E_n - E'_n). \quad (155)$$

The completeness relation (90) remains true, although with a modified factor D . Indeed, formula (91) as well as (101) and (103) change in so far as all factors of $|h_l(pR)|^2$ disappear. We will only need the completeness relation,

$$\tilde{D}^{-1} \int d\xi d\bar{\xi} |\psi_\xi\rangle \langle \psi_\xi| = I, \quad (156)$$

with

$$\tilde{D} = \int d\xi d\bar{\xi} \exp \left(- \int_{|E| \geq m} dE \sum_{l,m} \frac{p}{4\pi} |\xi_{l,m}(E)|^2 \right), \quad (157)$$

and the inner product,

$$\langle \psi_{(E_1, l_1, m_1), \dots, (E_n, l_n, m_n)} | \psi_\xi \rangle = \exp \left(- \int_{|E| \geq m} dE \sum_{l,m} \frac{p}{8\pi} |\xi_{l,m}(E)|^2 \right) \xi_{l_1, m_1}(E_1) \cdots \xi_{l_n, m_n}(E_n) \frac{p_1}{4\pi} \cdots \frac{p_n}{4\pi}. \quad (158)$$

To denote a state in $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ with q incoming particles with momenta k_1, \dots, k_q and $n - q$ outgoing particles with momenta k_{q+1}, \dots, k_n we write $\psi_{k_1, \dots, k_q | k_{q+1}, \dots, k_n}$. In the standard bra-ket notation this translates into,

$$\psi_{k_1, \dots, k_q | k_{q+1}, \dots, k_n} = |\psi_{k_1, \dots, k_q}\rangle \otimes \langle \psi_{k_{q+1}, \dots, k_n}|. \quad (159)$$

Let now $\hat{\xi}$ be a bounded classical solution in spacetime and $\hat{\xi}_+, \hat{\xi}_-$ the positive/negative frequency components as defined above. The associated coherent state in $\psi_\xi \in \mathcal{H}_1 \otimes \mathcal{H}_2^*$ takes in the standard bra-ket notation the form,

$$\psi_\xi = |\psi_{\xi_+}\rangle \otimes \langle \psi_{\xi_-}|. \quad (160)$$

The inner product of a coherent state ψ_ξ with an n -particle state $\psi_{k_1, \dots, k_q | k_{q+1}, \dots, k_n}$ thus takes the form

$$\langle \psi_\xi | \psi_{k_1, \dots, k_q | k_{q+1}, \dots, k_n} \rangle = \langle \psi_{\xi_+} | \psi_{k_1, \dots, k_q} \rangle \langle \psi_{k_{q+1}, \dots, k_n} | \psi_{\xi_-} \rangle = C_{\xi_+} C_{\xi_-} \overline{\xi_+(k_1)} \cdots \overline{\xi_+(k_q)} \overline{\xi_-(k_{q+1})} \cdots \overline{\xi_-(k_n)}, \quad (161)$$

where we have used (23). It will be useful to reexpress this in terms of the spherical harmonic modes of $\hat{\xi}$. To this end we notice that (39) implies,

$$\xi_\pm(k) = \int d^3x \left(E \hat{\xi}(x, t) \pm i \dot{\hat{\xi}}(x, t) \right) e^{\pm i(Et - kx)}, \quad (162)$$

where a dot indicates the derivative with respect to time. Inserting (152) and integrating over x yields,

$$\xi_\pm(k) = 2\pi \sum_{l,m} \xi_{l,m}(\mp E) i^{\mp l} Y_l^{-m}(\Omega_k), \quad (163)$$

where the angle coordinates Ω_k are given by the direction of the 3-vector k . Using this latter relation we rewrite (161) as,

$$\begin{aligned} \langle \psi_\xi | \psi_{k_1, \dots, k_q | k_{q+1}, \dots, k_n} \rangle &= (-1)^{l_{q+1} + \dots + l_n} (2\pi)^n i^{l_1 + \dots + l_n} \sum_{l_1, m_1} \cdots \sum_{l_n, m_n} \overline{\xi_{l_1, m_1}(-E_{k_1})} \cdots \overline{\xi_{l_q, m_q}(-E_{k_q})} \times \\ &\quad \overline{\xi_{l_{q+1}, m_{q+1}}(E_{k_{q+1}})} \cdots \overline{\xi_{l_n, m_n}(E_{k_n})} Y_{l_1}^{m_1}(\Omega_1) \cdots Y_{l_n}^{m_n}(\Omega_n) \exp \left(- \int_{|E| \geq m} dE \sum_{l,m} \frac{p}{8\pi} |\xi_{l,m}(E)|^2 \right). \end{aligned} \quad (164)$$

We now combine the inner products with the completeness relation (156) to an inner product between an n -particle state in the standard asymptotic state space $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ and an n -particle state in the hypercylinder asymptotic state space \mathcal{H}_{cyl} ,

$$\langle \psi_{(l_1, m_1, E_1), \dots, (l_n, m_n, E_n)} | \psi_{k_1, \dots, k_q | k_{q+1}, \dots, k_n} \rangle = \tilde{D}^{-1} \int d\xi d\bar{\xi} \langle \psi_{(l_1, m_1, E_1), \dots, (l_n, m_n, E_n)} | \psi_\xi \rangle \langle \psi_\xi | \psi_{k_1, \dots, k_q | k_{q+1}, \dots, k_n} \rangle. \quad (165)$$

We evaluate this using the expressions (157), (158) and (164) together with the usual method of shifting integration variables to arrive at,

$$\langle \psi_{(l_1, m_1, E_1), \dots, (l_n, m_n, E_n)} | \psi_{k_1, \dots, k_q | k_{q+1}, \dots, k_n} \rangle = (-1)^{l_{q+1} + \dots + l_n} (2\pi)^n i^{l_1 + \dots + l_n} Y_{l_1}^{m_1}(\Omega_{k_1}) \dots Y_{l_n}^{m_n}(\Omega_{k_n}) \times \\ \delta(E_1 + E_{k_1}) \dots \delta(E_q + E_{k_q}) \delta(E_{q+1} - E_{k_{q+1}}) \dots \delta(E_n - E_{k_n}). \quad (166)$$

We can now write an n -particle state in \mathcal{H}_{cyl} as a linear combination of n -particle states in $\mathcal{H}_1 \otimes \mathcal{H}_2^*$

$$\psi_{(l_1, m_1, E_1), \dots, (l_n, m_n, E_n)} = \frac{(-1)^{l_{q+1} + \dots + l_n} i^{-l_1 - \dots - l_n}}{(8\pi^2)^n} \int d\Omega_{p_1} \dots \int d\Omega_{p_n} Y_{l_1}^{-m_1}(\Omega_{p_1}) \dots Y_{l_n}^{-m_n}(\Omega_{p_n}) p_1 \dots p_n \times \\ \psi_{p_1, \dots, p_q | p_{q+1}, \dots, p_n}. \quad (167)$$

Reciprocally an n -particle state in $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ is a linear combination of n -particle states in \mathcal{H}_{cyl}

$$\psi_{k_1, \dots, k_q | k_{q+1}, \dots, k_n} = \frac{(8\pi^2)^n}{k_1 \dots k_n} \sum_{l_1, m_1} \dots \sum_{l_n, m_n} (-1)^{l_{q+1} + \dots + l_n} i^{l_1 + \dots + l_n} Y_{l_1}^{m_1}(\Omega_{k_1}) \dots Y_{l_n}^{m_n}(\Omega_{k_n}) \times \\ \psi_{(l_1, m_1, -E_{k_1}), \dots, (l_q, m_q, -E_{k_q}), (l_{q+1}, m_{q+1}, E_{k_{q+1}}), \dots, (l_n, m_n, E_{k_n})}. \quad (168)$$

The decomposition of (the finite radius version of) \mathcal{H}_{cyl} as a tensor product of in- and out-state spaces $\mathcal{H}^- \otimes \mathcal{H}^+$ was already introduced in [6]. Also, the fact that (with the present conventions) negative energy particles are in-particles while positive energy particles are out-particles was already discussed in that paper as well as in [5]. This already explains partially the formulas (167) and (168), especially with respect to the energy quantum numbers. From that perspective, what is new here is that we have constructed the isomorphisms $\mathcal{H}_1 \rightarrow \mathcal{H}^-$ between in-state spaces and $\mathcal{H}_2^* \rightarrow \mathcal{H}^+$ between out-state spaces. In terms of quantum numbers, this means we can now explicitly convert between the angular momentum quantum numbers l, m in \mathcal{H}_{cyl} and the three-momentum directions Ω_k in \mathcal{H}_1 and \mathcal{H}_2 .

VII. CONCLUSIONS AND OUTLOOK

The present result has immediate significance as a contribution to the GBF program in general, and to the extensibility conjecture specifically. It shows that and how perturbative interacting quantum field theory fits into the GBF for an asymptotic geometry that implements key non-standard features of the GBF. In particular, the geometry in question involves a region (the solid hypercylinder) whose boundary is timelike and connected, both features in contrast to what can be made sense of in the standard formalism. It is worth emphasizing at this point that the asymptotic geometry chosen is less special than it might seem from looking at its finite cousin. In contrast to the finite radius hypercylinder, its infinite radius limit is Poincaré invariant and may thus be seen as the limit of all possible timelike hypercylinders. This is analogous, of course, to what happens with the standard geometry. The asymptotic pair of equal-time hyperplanes at infinite initial and final time is Poincaré invariant and arises as the limit of all possible pairs of spacelike hyperplanes. The precise meaning of these statements is that the asymptotic amplitudes (58) and (150) are Poincaré invariant. This in turn comes from the fact that the propagator appearing in the underlying expressions (53) and (147) is Poincaré invariant. This reflects the Poincaré invariance of the relevant boundary conditions of the inhomogeneous Klein-Gordon equation. While this invariance is well known for the Feynman boundary conditions (50) it came as a surprise for the spatially asymptotic boundary conditions (138). Indeed, it follows from the even more surprising fact that both boundary conditions are equivalent (for a bounded source).

Concerning possible geometries for constructing asymptotic amplitudes, one might envision further ones than are different from both the standard one and the hypercylinder geometry. A fairly straightforward situation should be that obtained by parallel *timelike* hyperplanes [5, 6] and the limit of moving these to opposite spatial infinities. Based on our present finding we may say with confidence that the result would be an asymptotic amplitude equivalent to both the standard S-matrix as well as the asymptotic hypercylinder amplitude calculated here. A more interesting case would be to start with a finite region, e.g., a four-ball with its boundary three-sphere, letting the radius go to infinity. This would involve the novel feature of hypersurfaces that have both spacelike and timelike parts. A similarly interesting geometry could be that of a diamond. This would, apart from null hypersurfaces, involve *corners*. (See [4] for a discussion of corners in the GBF.) Of course, one should expect the asymptotic amplitudes constructed with these types of geometries to be equivalent as well if the extensibility conjecture is valid in some generality.

Apart from the relevance to the GBF program our result has further independent implications. One is related to the fact that there is only one spatial asymptotic state space in the hypercylinder geometry in contrast to the

two (initial and final) temporal asymptotic state spaces for the standard S-matrix. Whether a particle in the spatial asymptotic state space is in-coming or out-going (and hence appears under the isomorphism in the initial or final temporal asymptotic state space) is determined by its (energy) quantum numbers. Hence, the usual notion of crossing symmetry, a derived property in conventional treatments of the S-matrix, becomes a tautology in the hypercylinder geometry. To put it the other way round: If crossing symmetry did not hold, the GBF would be in trouble. Indeed, the anticipation of crossing symmetry as a manifest feature of the GBF served as an initial motivation for its development [13], independent from quantum gravity considerations.

Let us explore further the new spatially asymptotic amplitude introduced in this paper. We have seen in Section VI that it is equivalent to the usual S-matrix (when both make sense). What is more, as shown in [2, 6] the GBF gives it a full fledged physical interpretation, fit for its description of scattering processes, and independent of the mentioned equivalence. Hence, the spatial asymptotic amplitude may be called “S-matrix” with the same justification as the usual one and we will do so from here onwards. However, this does not mean that both S-matrices are equally applicable in all physical situations. Indeed, for the usual S-matrix to make sense, interactions must be negligible at very early and at very late times. In contrast, the spatially asymptotic S-matrix requires that interactions are negligible at large distance from a center. However, the interactions may remain important at all times. One may argue that the latter restriction is more naturally met than the former in many physical situations of interest, such as (almost) stationary processes. One may thus expect a useful role for the new S-matrix for the description of certain processes where the usual S-matrix approach fails.

Generalizing our approach to curved spacetime, the difference between the new S-matrix and the usual one would become even more important. What we mean here with this generalization is the following. One would choose in the spacetime in question a region that shares the characteristic features of the solid hypercylinder (such as its topology and the fact that its boundary is timelike). One would then establish the interaction picture for such hypercylinders under radial “evolution”. Finally, one would compute the asymptotic amplitude when the radius is taken to infinity. Of course, this would work only in a certain class of spacetimes. For example, a condition would be that space be non-compact. The point is that the class of spacetimes where this would work is different from the one where a conventional S-matrix description works. For example, the present approach should be applicable to Anti-de Sitter space, where a conventional S-matrix does not exist because a useful notion of temporal asymptotic state space is lacking. Nevertheless, an “S-matrix analogue” (which plays an important role in the conjectured AdS/CFT correspondence [14]) has been constructed for Anti-de Sitter in a more heuristic way [15]. The present approach should likely lead at least to a better conceptual underpinning of this construction. Another interesting example could be a stationary black hole spacetime. Placing a hypercylinder carrying the states outside the horizon naturally avoids having to say anything about the black hole interior within this state space. Unsurprisingly, ideas in this direction are already implicit in approaches to black hole physics. For example, ’t Hooft factorizes a hypothetical black hole S-matrix into three pieces [16], one of which could be interpreted as corresponding to a hypercylinder amplitude just outside the horizon.

The method of derivation used here for the new as well as for the standard S-matrix merits an additional comment. In contrast to most treatments of the S-matrix we first construct finite interacting (transition) amplitudes explicitly and then take the respective asymptotic limit. This is facilitated by the use of coherent states. For the case of the hypercylinder geometry these were not known previously, so their introduction in Section IV B constitutes a side result of this paper.

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APPENDIX A: TRANSFORMATION OF COHERENT STATES INTO THE SCHRÖDINGER REPRESENTATION AND POLYNOMIAL REPRESENTATION

In order to determine the wave functions of coherent states in the Schrödinger representation it is useful to transform to a representation which we shall call the *polynomial representation*. This representation is related to the holomorphic one, although we will not discuss this relation here.

We first consider the context of equal-time hyperplanes. Recall that the wave function of a multi-particle state in the Schrödinger representation is a product of a polynomial with the vacuum wave function. Say, we have n particles with momenta p_1, \dots, p_n . The wave function of the associated normalized state takes the form

$$\psi_{p_1, \dots, p_n}(\varphi) = P_{p_1, \dots, p_n}(\varphi) \psi_0(\varphi). \quad (\text{A1})$$

P_{p_1, \dots, p_n} is a polynomial of order n in the field configuration φ . It is not a monomial, however, but rather a linear combination of monomials of orders $n, n-2, n-4$ etc. The order n component is the product $\check{\varphi}(p_1) \cdots \check{\varphi}(p_n)$ (in the conventions of [6]) where $\check{\varphi}(p)$ is the functional given by Fourier transform,

$$\check{\varphi}(p) := 2E \int d^3x e^{ipx} \varphi(x). \quad (\text{A2})$$

As the Schrödinger representation, the representation we are going to define consists of a certain space of functions on instantaneous field configurations. We can thus think of its elements also as “wave functions”. To distinguish them from the wave functions in the Schrödinger representation we add a tilde to the former. Concretely, the transformation $\psi \mapsto \tilde{\psi}$ between the representations is given by the formula,

$$\tilde{\psi}(\varphi) := |C|^2 \int \mathcal{D}\varphi' \frac{\psi(\varphi + \varphi')}{\psi_0(\varphi + \varphi')} \exp \left(- \int d^3x \varphi'(x) (\omega \varphi')(x) \right), \quad (\text{A3})$$

where C is the usual normalization constant appearing in the vacuum wave function (15). Note that the inverse transform is given by,

$$\psi(\varphi) = |\tilde{C}|^2 \psi_0(\varphi) \int \mathcal{D}\varphi' \tilde{\psi}(\varphi + \varphi') \exp \left(\int d^3x \varphi'(x) (\omega \varphi')(x) \right), \quad (\text{A4})$$

where the normalization factor \tilde{C} is formally defined via

$$|\tilde{C}|^{-2} := \int \mathcal{D}\varphi \exp \left(\int d^3x \varphi(x) (\omega \varphi)(x) \right). \quad (\text{A5})$$

The defined representation, to which we shall refer as the *polynomial representation*, has the nice property that the wave function of an n -particle state is simply a monomial of degree n ,

$$\tilde{\psi}_{p_1, \dots, p_n}(\varphi) = \check{\varphi}(p_1) \cdots \check{\varphi}(p_n). \quad (\text{A6})$$

The vacuum wave function is just the unit constant, $\tilde{\psi}_0(\varphi) = 1$. A creation operator acts by multiplication, while an annihilation operator acts by derivation,

$$(a^\dagger(p) \tilde{\psi})(\varphi) = \check{\varphi}(p) \tilde{\psi}(\varphi), \quad (a(p) \tilde{\psi})(\varphi) = \int d^3x e^{-ipx} \frac{\delta}{\delta \varphi(x)} \tilde{\psi}(\varphi). \quad (\text{A7})$$

The simple action of the creation operator makes it easy to work out the wave function of the coherent state (16) of Section IIB in the polynomial representation. We obviously get,

$$\tilde{\psi}_\eta(\varphi) = C_\eta \exp \left(\int \frac{d^3k}{(2\pi)^{32E}} \eta(k) \check{\varphi}(k) \right). \quad (\text{A8})$$

It remains to apply the inverse transform (A4) to obtain the Schrödinger representation wave function. We can read off the field dependent part immediately: Inserting a sum of two fields into the exponential simply gives a product of exponentials. So, the field dependence remains exactly the same in the Schrödinger representation (up to the vacuum wave function factor),

$$\psi_\eta(\varphi) = K_\eta \exp \left(\int \frac{d^3k}{(2\pi)^{32E}} \eta(k) \check{\varphi}(k) \right) \psi_0(\varphi). \quad (\text{A9})$$

Only the η -dependent normalization factor changes. It can be computed with the usual method of shifting integration variables and the result is given by (25).

The Schrödinger representation on the hypercylinder has a structure analogous to the one on equal-time hyperplanes [6]. Indeed, formulas are structurally identical. This is also true for the polynomial representation and its relation to the Schrödinger representation. We mention here only the transformation formula analogous to (A3) which now reads,

$$\tilde{\psi}(\varphi) := |C_R|^2 \int \mathcal{D}\varphi' \frac{\psi(\varphi + \varphi')}{\psi_{R,0}(\varphi + \varphi')} \exp \left(- \int_{|E| \geq m} dE \sum_{l,m} \varphi'_{l,m}(E) \frac{2\pi}{p|h_l(pR)|^2} \varphi'_{l,-m}(-E) \right), \quad (\text{A10})$$

where C_R is the usual normalization constant appearing in the vacuum wave function (79). Hence, if we define a coherent state on the hypercylinder via creation operators acting on the vacuum in analogy to (16) the result will necessarily be of a form analogous to (A9). This may serve as a justification for the definition (104) in Section IV B.

APPENDIX B: SPHERICAL HARMONICS AND SPHERICAL BESSEL FUNCTIONS

The solutions of the eigenvalue problem corresponding to the angular part of the Klein-Gordon equation in spherical coordinates, namely equation (63), are given by the spherical harmonics denoted Y_l^m . These functions are defined through the associated Legendre function P_l^m [17] via

$$Y_l^m(\theta, \phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos \theta) e^{im\phi}. \quad (\text{B1})$$

and satisfy the orthogonality relation

$$\int d\Omega Y_l^m \overline{Y_{l'}^{m'}} = \delta_{l,l'} \delta_{m,m'}, \quad (\text{B2})$$

where $\int d\Omega = 4\pi$.⁴ Note also, $\overline{Y_l^m} = Y_l^{-m}$.

The radial part of the Klein-Gordon equation in spherical coordinates, namely equation (64), is solved by the so called *spherical Bessel functions* of the first kind j_l and of the second kind n_l and the *modified spherical Bessel functions* of the first kind i_l^+ and of the second kind i_l^- .

The spherical Bessel functions j_l and n_l can be expressed in terms of the ordinary Bessel functions of the first and second kind, J_l and N_l respectively, as

$$j_l(z) = \sqrt{\frac{\pi}{2z}} J_{l+1/2}(z), \quad \text{and} \quad n_l(z) = \sqrt{\frac{\pi}{2z}} N_{l+1/2}(z). \quad (\text{B3})$$

Here we use the conventions of [17] for the ordinary spherical Bessel functions, *but* different conventions for the modified spherical Bessel functions. Indeed we define the modified spherical Bessel functions as the analytic continuation of the ordinary ones,

$$i_l^+(z) := j_l(e^{i\pi/2}z), \quad \text{and} \quad i_l^-(z) := n_l(e^{i\pi/2}z). \quad (\text{B4})$$

Our convention differs by powers of i from the definitions 10.2.2 and 10.2.3 of [17]. However, this turns out to be more convenient for our treatment. We note that i_l^+ and i_l^- are always either real or imaginary (depending on l) and the product $i_l^+ i_l^-$ is always imaginary.

The Bessel functions satisfy the orthogonality relation

$$\int_0^\infty dz z J_l(\alpha z) J_l(\beta z) = \frac{1}{\alpha} \delta(\alpha - \beta), \quad \text{for} \quad l > -1, \alpha > 0, \beta > 0. \quad (\text{B5})$$

An analogous relation is valid for N_l . (B5) follows from formulas 6.633.2 of [18] and 9.7.1 of [17]. This relation implies for the spherical Bessel functions,

$$\int_0^\infty dz z^2 j_l(\alpha z) j_l(\beta z) = \frac{\pi}{2\alpha^2} \delta(\alpha - \beta), \quad \text{for} \quad l > -\frac{3}{2}, \alpha > 0, \beta > 0. \quad (\text{B6})$$

The same relation holds for n_l .

The Wronskians for the spherical Bessel functions and the modified spherical Bessel functions are

$$j_l(z) n_l'(z) - j_l'(z) n_l(z) = \frac{1}{z^2}, \quad \text{and} \quad i_l^+(z) (i_l^-)'(z) - (i_l^+)'(z) i_l^-(z) = -\frac{i}{z^2}. \quad (\text{B7})$$

where the first equation above corresponds to 10.1.6 of [17] and the second follows from (B4) and 10.2.7 of [17]. Following the generalized notation for the ordinary and the modified spherical Bessel functions introduced at the end of Section IV A 1, formulas (68), from (B7) we derive the relation

$$a_l(E, z) b_l'(E, z) - a_l'(E, z) b_l(E, z) = \frac{1}{pz^2}, \quad \text{with} \quad p := \begin{cases} \sqrt{E^2 - m^2} & \text{if } E^2 > m^2, \\ i\sqrt{m^2 - E^2} & \text{if } E^2 < m^2. \end{cases} \quad (\text{B8})$$

⁴ We differ here slightly from the conventions in [6], where $\int d\Omega = 1$.

The partial wave decomposition of a plane wave is expressed in terms of spherical harmonics and spherical Bessel functions as (see formula (B.105) of [19])

$$e^{ikz} = 4\pi \sum_{l,m} i^l j_l(E_k|z|) \overline{Y_l^m(\Omega_k)} Y_l^m(\Omega_z), \quad (\text{B9})$$

where Ω_k and Ω_z represent the θ, ϕ directions of the 3-vectors k and z respectively, and $E_k = \sqrt{|k|^2 + m^2}$. Using the notation (68) for the spherical Bessel functions, from the above decomposition of a plane wave follows the relation

$$\frac{e^{ip|z-z'|}}{4\pi|z-z'|} = ip \sum_{l,m} \overline{Y_l^m(\Omega_z)} Y_l^m(\Omega_{z'}) \{ \theta(|z| - |z'|) c_l(E, |z|) a_l(E, |z|) + \theta(|z'| - |z|) c_l(E, |z'|) a_l(E, |z'|) \}, \quad (\text{B10})$$

see formulas (B.98) and (B.100) of [19]. Notice that p can be real or imaginary.

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